ITERATIVE METHODS FOR PAIRS OF ADDITIVE DIOPHANTINE EQUATIONS

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ABSTRACT. We outline an approach, based on the Hardy-Littlewood method, for demonstrating that a pair of additive equations has a non-trivial integral solution. Particular attention is given to the issues surrounding an optimal implementation of Wooley's iterative method for estimating mean values of exponential sums.

1. INTRODUCTION

Let k and n be integers with $k \geq n \geq 1$, and let $c_1, \dots, c_s$ and $d_1, \dots, d_s$ be non-zero integers. We consider the problem of determining conditions under which one can demonstrate that the system of equations

$$\begin{aligned} c_1 x_1^k + \cdots + c_s x_s^k &= 0 \\ d_1 x_1^n + \cdots + d_s x_s^n &= 0 \end{aligned} \tag{1}$$

possesses a non-trivial integral solution. One obvious requirement is that the system must have a non-trivial real solution and a non-trivial p-adic solution for every prime p. In fact, the success of the Hardy-Littlewood method depends explicitly upon good information concerning the density of such solutions, and hence one must typically show that there are non-singular solutions in each local field. Unfortunately, the local solubility problem for (1) tends to be quite hard. When $p > k^4 n^2$, Wooley [25] demonstrates the existence of non-trivial p-adic solutions to (1) provided only that $s > 2(k+n)$, but the number of variables required for smaller primes in the recent work of Knapp [14] is often significantly larger than what would be required to handle the minor arcs in an application of the circle method. Moreover, it is difficult to guarantee that the solutions produced by the p-normalization methods of [14] and [25] are non-singular when $k - n > 1$. We therefore focus our attention on determining how large s must be in order to establish a local-global principle for (1). Define $G^*(k, n)$ to be the least integer r such that, whenever $s \geq r$ and the system

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1991 *Mathematics Subject Classification.* 11D72, 11P55.

(1) has a non-singular real solution and a non-singular p-adic solution for every prime p, the system (1) has a non-trivial integral solution.

The situation in which $k = n$ has attracted interest for quite some time, beginning with work of Davenport and Lewis [11] on pairs of cubics in 18 variables. Cook [9] showed that $G^*(2,2) \le 9$, while the bound of Davenport and Lewis for $G^*(3,3)$ has been steadily reduced over the years by work of Cook [10], Vaughan [19], Baker and Brüdern [4], and finally Brüdern [7], who obtained $G^*(3,3) \le 14$. Obstructions to local solubility are already apparent here, as [11] shows that there are pairs of cubics in 15 variables possessing no non-trivial 7-adic solution. In attempting to bound $G^*(k,k)$, one considers exponential sums of the type

$$F(\gamma) = \sum_{|x| \le P} e(\gamma x^k),$$

where we have written $e(z) = e^{2\pi i z}$. In practice, the summation is often further restricted to integers that are free of large prime factors, but we suppress this point for now. One then observes that the number of solutions to (1) lying in the box $[-P,P]^s$ is given by the integral

$$\int_{[0,1]^2} \prod_{i=1}^{s} F(c_i\alpha + d_i\beta)\, d\alpha\, d\beta.$$

To obtain an upper bound for the minor arc contribution to this integral, one can apply Hölder's inequality and then make a change of variables that factors the integral into a product of one-dimensional mean values to which the estimates of Vaughan and Wooley [22], [24] apply. Thus one typically expects bounds for $G^*(k,k)$ that are about twice the size of the corresponding bound for $G(k)$ in Waring's problem. However, the sharpest bounds for $G(k)$ are often obtained by employing a form of p-adic iteration to save an extra variable over what would result from a direct application of the mean value estimates and Weyl's inequality. As illustrated in Brüdern [7], the extension of such techniques to pairs of equations can present serious technical challenges, although the argument of [18], leading to the bound $G^*(5,5) \le 34$, is actually somewhat manageable. For larger k, the methods of Brüdern and Cook [8] show that $G^*(k,k) \le (2 + o(1))k \log k$, which compares as expected with the well-known bound of Wooley [28].

Pairs of equations of differing degree have received considerably less attention, as their study requires a distinctly two-dimensional approach involving exponential sums of the shape

$$F(\alpha,\beta) = \sum_{|x| \le P} e(\alpha x^k + \beta x^n).$$

This situation was first tackled by Wooley [27], who developed a version of Vaughan's iterative method [20] suitable for generating mean value estimates for sums of this type restricted to smooth numbers. These estimates initially produced the bound $G^*(3,2) \leq 14$, which was later improved to $G^*(3,2) \leq 13$ in [30] using the results of [29]. In this particular case, Wooley [26] was actually able to establish p-adic solubility whenever $s \geq 11$, and hence one only needs to impose a real solubility hypothesis.

It is a straightforward exercise to generate bounds for other pairs of exponents using the general method of Wooley [29]. In particular, one should be able to demonstrate with little difficulty that $G^*(k,n) \leq (2+o(1))k \log k$, for large $k > n$. However, as in the current treatment of Waring's problem (see Vaughan and Wooley [22], [24]), there are various refinements that may be attempted in order to obtain good results for smaller exponents. As the amount of available technology associated with the Hardy-Littlewood method is nowadays quite substantial, one has many options for carrying out such refinements, and it is a non-trivial task to determine the optimal strategy in each case. Our goal here is mainly to provide an overview of the various possible approaches and their limitations, so we defer most of the technical details to [17]. In the following table, the entry appearing in row k and column n is the upper bound we obtain for $G^*(k,n)$.

	1	2	3	4	5	6	7
3	10	13	14				
4	17	20	24	24			
5	30	31	32	36	34		
6	49	50	49	47	50	49	
7	66	72	70	65	64	66	67

As mentioned above, the estimates for $G^*(3,3)$, $G^*(5,5)$, and $G^*(3,2)$ are obtained from [7], [18], and [30], respectively. The bounds for $G^*(k,k)$ in the remaining cases follow in relatively routine fashion from the mean value estimates of [20], [22], and [24], and it may be possible to save an additional variable in some of these instances by proceeding as in [18].

The estimates for mean values of exponential sums obtained in our analysis may be further applied to deal with the corresponding problem on pairs of diophantine inequalities, where one seeks to demonstrate that two forms with real coefficients take arbitrarily small values simultaneously at integral points. This problem has already been investigated by the author [15], [16] in the case of a cubic and quadratic form by employing the mean value estimates of Wooley [30], together with some new ideas of Bentkus and Götze [5] and Freeman [12]. We intend to pursue this application for other pairs of exponents in a later paper.

2. The Analytic Set-up

After possibly replacing some of the variables x_i by $-x_i$ and then changing the signs of the corresponding coefficients in forms of odd degree, we may suppose that the system (1) has a non-singular real solution with all coordinates positive, and hence it suffices to consider solubility in positive integers. Let

$$\mathcal{A}(P,R) = \{n \in [1,P] \cap \mathbb{Z} : p|n \Rightarrow p \le R\}$$

denote the set of R-smooth numbers of size at most P. As usual, we take P to be a large positive number and R to be a small positive power of P, so that one has $\mathrm{card}(\mathcal{A}(P,R)) \gg P$ (see for example Vaughan [21], §12.1). We now write $\boldsymbol{\alpha} = (\alpha, \beta)$ and define the exponential sums

$$F(\boldsymbol{\alpha}) = \sum_{1 \le x \le P} e(\alpha x^k + \beta x^n)$$

and

$$f(\boldsymbol{\alpha}) = \sum_{x \in \mathcal{A}(P,R)} e(\alpha x^k + \beta x^n). \tag{2}$$

Further, write $F_i(\boldsymbol{\alpha}) = F(c_i\alpha, d_i\beta)$ and $f_i(\boldsymbol{\alpha}) = f(c_i\alpha, d_i\beta)$, and introduce the decomposition $s = t + 2u$. Then one sees that

$$N(P) = \int_{[0,1]^2} \prod_{i=1}^{t} F_i(\boldsymbol{\alpha}) \prod_{i=t+1}^{s} f_i(\boldsymbol{\alpha})\, d\boldsymbol{\alpha}$$

is the number of solutions of (1) with the variables satisfying

$$1 \le x_i \le P \quad (i = 1, \dots, t) \quad \text{and} \quad x_i \in \mathcal{A}(P,R) \quad (i = t+1, \dots, s).$$

We now describe our Hardy-Littlewood dissection. Write $t_k = \max |c_i|$ and $t_n = \max |d_i|$, and let $X_i = 2k^2 t_i P^{i-1}$ for $i = k, n$. Define the major arcs $\mathfrak{M}$ to be the union of the rectangles

$$\mathfrak{M}(q,a,b) = \{\boldsymbol{\alpha} \in [0,1)^2 : |q\alpha - a| \le X_k^{-1} \text{ and } |q\beta - b| \le X_n^{-1}\}$$

with $0 \le a, b \le q \le P$ and $(q,a,b) = 1$, and write

$$\mathfrak{m} = [0,1]^2 \setminus \mathfrak{M}$$

for the minor arcs. Provided that s and t are not too small, the existence of non-singular real and p-adic solutions allows one to show that

$$\int_{\mathfrak{M}} \prod_{i=1}^{t} F_i(\boldsymbol{\alpha}) \prod_{i=t+1}^{s} f_i(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \gg P^{s-(k+n)} \tag{3}$$

by employing a straightforward extension of the argument of Wooley [27]. Here one uses the t variables ranging over a complete interval to prune back to a thinner set of major arcs, on which asymptotics for $f_i(\boldsymbol{\alpha})$ can be obtained. For the minor arcs, an application of Hölder's inequality, together with a consideration of the underlying diophantine equations, shows that

$$\int_{\mathfrak{m}} \prod_{i=1}^{t} F_i(\boldsymbol{\alpha}) \prod_{i=t+1}^{s} f_i(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \ll \sup_{\boldsymbol{\alpha}\in\mathfrak{m}} |F_i(\boldsymbol{\alpha})|^t \int_{[0,1]^2} |f(\boldsymbol{\alpha})|^{2u}\, d\boldsymbol{\alpha}$$

for some i. A minor arc bound for $F_i(\boldsymbol{\alpha})$ is provided by a generalization of Weyl's inequality due to Baker [1] (see also [2], [3]), while the estimation of the even moments of $f(\boldsymbol{\alpha})$ represents our main challenge. Baker's result allows us to save essentially P^{σ} per variable over the trivial estimate, where $\sigma = 2^{1-k}$, and our mean value estimates take the shape

$$\int_{[0,1]^2} |f(\boldsymbol{\alpha})|^{2u} d\boldsymbol{\alpha} \ll P^{2u-(k+n)+\Delta_u+\varepsilon}, \tag{4}$$

where an estimate with $\Delta_u = 0$ would be essentially best possible. We therefore obtain the bound

$$\int_{\mathfrak{m}} \prod_{i=1}^{t} F_i(\boldsymbol{\alpha}) \prod_{i=t+1}^{s} f_i(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \ll P^{s-(k+n)-\delta}$$

for some $\delta > 0$, provided that $\sigma t > \Delta_u$. Thus on recalling (3), one has

$$G^*(k,n) \le \min_{u\in\mathbb{N}} \left(2u + \left[\frac{\Delta_u}{\sigma}\right] + 1\right). \tag{5}$$

We concentrate on obtaining bounds for Δ_u in the remaining sections.

3. Efficient Differencing

One sees from the above discussion that the number of variables required to establish a local-global principle for the system (1) is closely connected with the strength of the available estimates for mean values of the exponential sums (2). We now indicate a strategy, based on the method of Wooley [29], for obtaining such mean value estimates. Wooley [27] showed using elementary methods that the number of solutions of the system

$$\begin{aligned} x_1^k + x_2^k + x_3^k &= y_1^k + y_2^k + y_3^k \\ x_1^n + x_2^n + x_3^n &= y_1^n + y_2^n + y_3^n \end{aligned} \tag{6}$$

with $x_i, y_i \in [1,P]$ is $O(P^{3+\varepsilon})$ whenever $k > n \ge 1$. This provides an estimate for the 6th moment of $F(\boldsymbol{\alpha})$, and hence of $f(\boldsymbol{\alpha})$, that is essentially best possible in view of the diagonal solutions to (6). One only saves P^3 over the trivial estimate, however, so in the context of (4) one is forced to take $\Delta_3 = k + n - 3$. An examination of (5) with $u = 3$ therefore produces unimpressive bounds such as $G^*(3,2) \le 15$ and $G^*(5,3) \le 87$. To improve on these, one instead uses the information concerning the number of solutions to (6) as the basis for an iteration to higher moments via the method of Vaughan [20] and Wooley [28], [29].

A typical step in the iteration may be summarized as follows. Suppose that one has an estimate of the shape (4) for the mean value

$$S_s(P,R) = \int\limits_{[0,1]^2} |f(\boldsymbol{\alpha})|^{2s}\, d\boldsymbol{\alpha},$$

and write $\lambda_s = 2s - (k+n) + \Delta_s$. Then $S_{s+2}(P,R)$ is bounded above by the number of solutions of the system

$$\begin{aligned} z_1^k - w_1^k + z_2^k - w_2^k &= \sum_{i=1}^{s} (x_i^k - y_i^k) \\ z_1^n - w_1^n + z_2^n - w_2^n &= \sum_{i=1}^{s} (x_i^n - y_i^n) \end{aligned} \tag{7}$$

with $x_i, y_i \in \mathcal{A}(P,R)$ and $z_i, w_i \in [1,P]$. Now let $\theta \leq 1/k$ be a parameter at our disposal, and write

$$M = P^\theta, \quad Q = PM^{-1}, \quad \text{and} \quad H = PM^{-k}.$$

The solutions of (7) with some x_i or y_i smaller than M can be shown to contribute a negligible amount, and for the remaining solutions the fact that x_i and y_i are R-smooth implies that each has a divisor lying between M and MR. Then by applying Hölder's inequality, one reduces to the situation in which the divisors are all identical and is thus led to analyze the system

$$\begin{aligned} z_1^k - w_1^k + z_2^k - w_2^k &= m^k \sum_{i=1}^{s} (u_i^k - v_i^k) \\ z_1^n - w_1^n + z_2^n - w_2^n &= m^n \sum_{i=1}^{s} (u_i^n - v_i^n) \end{aligned} \tag{8}$$

with $u_i, v_i \in \mathcal{A}(Q,R)$, $M < m \leq MR$, and $z_i, w_i \in [1,P]$. One may further suppose, after some effort, that m is coprime to any Jacobian determinant of z_i and w_j arising from the left-hand side.

Equipped with an estimate for the number of non-singular solutions in z_i, w_i, distinct modulo m^k, to the implicit pair of congruences in (8), one can further reduce, via Cauchy's inequality, to the situation in which

$$z_1 \equiv w_1 \pmod{m^k} \quad \text{and} \quad z_2 \equiv w_2 \pmod{m^k}. \tag{9}$$

These congruence conditions allow us to perform a differencing operation that is more efficient than the classical one of Weyl, since we can now write

$$w_1 = z_1 + h_1 m^k \quad \text{and} \quad w_2 = z_2 + h_2 m^k$$

with h_1 and h_2 of magnitude at most H. It is actually convenient to write $x_i = z_i + w_i$ and to consider the symmetric difference polynomials

$$\psi_i(x, h, m) = m^{-i}((x + hm^k)^i - (x - hm^k)^i).$$

One then finds (essentially) that

$$S_{s+2}(P, R) \ll P^\varepsilon M^{2s+k-n-1} \int\limits_{[0,1]^2} \mathcal{F}_1(\boldsymbol{\alpha}) |f_1(\boldsymbol{\alpha})|^{2s} \, d\boldsymbol{\alpha}, \tag{10}$$

where

$$\mathcal{F}_1(\boldsymbol{\alpha}) = \sum_m \left| \sum_{h,z} e(\alpha_k \psi_k(z, h, m) + \alpha_n \psi_n(z, h, m)) \right|^2,$$

with the summations running over

$$M < m \le MR, \quad 1 \le h \le H, \quad \text{and} \quad 1 \le z \le 2P,$$

and where

$$f_1(\boldsymbol{\alpha}) = \sum_{x \in \mathcal{A}(2Q,R)} e(\alpha_k x^k + \alpha_n x^n).$$

Of course, one also needs to account for the terms with $h = 0$, but in practice one usually chooses θ in such a way that this diagonal contribution is of the same order of magnitude as the expression in (10). The factor of $M^{2s+k-n-1}$ appearing in (10) represents the cost of obtaining the uniform divisor m and imposing the strong congruence condition (9). One can now obtain an estimate for $S_{s+2}(P, R)$ by employing the trivial bound $\mathcal{F}_1 \ll MH^2P^2$ together with the estimate $S_s(2Q, R) \ll Q^{\lambda_s + \varepsilon}$, which was obtained at the previous stage. Alternatively, one could extract divisors from the u_i and v_i in (8) and then repeat the differencing argument, perhaps multiple times, before making a final estimate. More creative strategies for dealing with (10), along the lines of Vaughan and Wooley [22], [24], are described in the next section.

4. End-game Strategies

If we apply the efficient differencing argument of the previous section j times, then we choose parameters $M_1, \ldots, M_j$ corresponding to the sizes of the divisors extracted at each difference. The variables u_i and v_i in the analogue of (8) are represented by an exponential sum f_j that is identical to f_1 except that the variables range only up to $2^j P(M_1 \cdots M_j)^{-1}$. The polynomial ψ_i is replaced by a polynomial of degree $i-j$ in z that is also a function of the divisors $m_1, \ldots, m_j$ and difference parameters $h_1, \ldots, h_j$. We can then write down an exponential sum $\mathcal{F}_j$ analogous to $\mathcal{F}_1$ and attempt to estimate an integral of the shape

$$\int_{[0,1]^2} \mathcal{F}_j(\boldsymbol{\alpha})|f_j(\boldsymbol{\alpha})|^{2r}\, d\boldsymbol{\alpha}. \tag{11}$$

Here r is typically about $s-j$, where s is as in (10). Finally, we trace this estimate back through the differencing process, optimizing the parameters M_i along the way, until we reach (10). The fundamental decisions to be made are how many differences to take and how to estimate the integral (11) after the final difference.

One possible approach, useful near the beginning of the iteration, is to consider estimates for the various moments of $\mathcal{F}_j(\boldsymbol{\alpha})$. For example, it is fairly easy to demonstrate by considering the underlying diophantine equations that one has

$$\int_{[0,1]^2} \mathcal{F}_1(\boldsymbol{\alpha})^2\, d\boldsymbol{\alpha} \ll P^{2+\varepsilon} M^2 H^3, \tag{12}$$

with similar results holding for $\mathcal{F}_j$ whenever $j \leq k-2$. Higher moments can also be estimated, but the bounds are in most cases too weak to be useful in light of the other available methods. In order to use an estimate such as (12), we apply Hölder's inequality to (11), and this in turn may require us to estimate a higher moment of f_j that was not scheduled to be dealt with until later in the iteration. Thus we should first obtain preliminary bounds for all the relevant moments by applying (for example) the method described at the end of §3. It is sometimes effective when using mean values to employ a generalization of the differencing argument in which the congruence condition (9) is weakened in order to reduce the cost of imposing it. In bounding $G^*(5,3)$, for example, it is useful to consider the congruences modulo m_j^3 rather than modulo m_j^5 on the final difference when s is small.

Towards the end of the iteration, the use of mean values tends to become less effective. If we are sufficiently far from a diagonal situation, however, it is often possible to obtain major and minor arc estimates for the sums $\mathcal{F}_j$ and hence to treat the integral (11) by means of a Hardy-Littlewood dissection. For example, in the case $k=5$ and $n=3$ one has

$$\alpha_5\psi_5 + \alpha_3\psi_3 = 10h\alpha_5 z^4 + (20h^3m^{10}\alpha_5 + 6hm^2\alpha_3)z^2,$$

and one can deduce from Baker [1] that there are good rational approximations to the coefficients of z^4 and z^2 above (for many values of h and m)

whenever the sum $\mathcal{F}_1$ is unusually large. However, the resulting approximations to α_5 and α_3 certainly depend on h and m, and it is not immediately clear that there is any fixed choice for which the approximations are of major arc quality. Fortunately, Baker's argument confronts essentially the same issue, in the context of ordinary Weyl differencing, and employs a result of Birch and Davenport [6] to obtain uniform approximations of the desired quality. While a version of Baker's argument works in principle for our situation, there are in fact several conditions to check, and these tend to fail unless j is relatively small in terms of k and n. In the case $k = 5$ and $n = 3$, for example, the method is currently only successful for first differences. A possible alternative for larger j is to attempt a one-dimensional dissection based solely on approximations to α_k, but this is usually inferior to the other available methods.

Finally, the constant interplay between exponential sums and underlying diophantine equations gives rise to a variety of ad-hoc methods for estimating the integral (11) more directly. For example, if we difference n times, then we reduce to a system analogous to (8), but where the left-hand side of the second equation is zero. For the quintic-cubic case, the resulting system is

$$480h_1h_2h_3(z_1^2 - z_2^2) = \sum_{i=1}^{r}(u_i^5 - v_i^5), \qquad 0 = \sum_{i=1}^{r}(u_i^3 - v_i^3).$$

To count the solutions with $z_1 = z_2$, it suffices to fix the variables z_1, h_1, h_2, and h_3 and to apply an estimate for $S_r(P, R)$. If $z_1 \neq z_2$, then we can instead fix the u_i and v_i using the second equation, at which point h_1, h_2, h_3, z_1, and z_2 are determined to $O(P^\varepsilon)$ by a divisor estimate. Note that in either case the variables m_1, m_2, and m_3 must be accounted for by a trivial estimate, since they do not appear explicitly in the system. This method turns out to be quite effective in the intermediate stages of the iteration and is noteworthy because it has no analogue in a one-dimensional situation like Waring's problem. When $k-n = 1$ and $j \leq n-1$, a variation employed by Wooley in bounding $G^*(3, 2)$ is often somewhat more effective. Here one deals with the non-diagonal solutions by taking a linear combination of the two equations and then applying a generalization of Hua's inequality ([13], Theorem 4). It is in some ways natural to view the type of analysis described in this paragraph as a more direct version of a Hardy-Littlewood dissection, with the non-diagonal and diagonal solutions corresponding respectively to the major and minor arcs (see [23]).

The whole iterative process can now be repeated, seeded with improved preliminary estimates, until the values of λ_s stabilize. Because of the number of possibilities at each stage, we use a computer to determine the optimal iterative scheme for each pair of exponents. In the quintic-cubic case, a combination of mean values, dissections, and ad-hoc analyses permits us to obtain $\Delta_{15} \leq 0.11321$, and the bound $G^*(5,3) \leq 32$ then follows immediately from (5).

It is generally not effective to difference more than n times, because the loss of the non-singularity condition on the implicit pair of congruences would then essentially force us into a one-dimensional analysis. It is therefore difficult to generate good estimates for $G^*(k,n)$ when n is small relative to k, since a large value of k would ordinarily call for a large number of differences. As a consequence, we sometimes obtain better bounds for $G^*(k,n)$ when n is close to k than for $G^*(k,1)$ or $G^*(k,2)$. This is contrary to the expectation that systems of lower total degree should require fewer variables to solve, but it seems to be a fundamental difficulty associated with the method.

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