DISSECTION OF THE QUINTUPLE PRODUCT, WITH APPLICATIONS

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Abstract. This work considers the m-dissection (for $m \not\equiv 0 \pmod 3$) of the general quintuple product

$$Q(z,q) = (z, q/z, q; q)_\infty (qz^2, q/z^2; q^2)_\infty.$$

Multiple novel applications arise from this m-dissection. For example, we derive the general partition identity

$$D_S(mn + (m^2-1)/24) = (-1)^{(m+1)/6} b_m(n), \qquad \text{for all } n \geq 0,$$

where $m \equiv 5 \pmod 6$ is a square-free positive integer relatively prime to 6; $D_S(n)$ is defined, for S the set of positive integers containing no multiples of m, to be the number of partitions of n into an even number of distinct parts from S minus the number of partitions of n into an odd number of distinct parts from S; and $b_m(n)$ denotes the number of m-regular partitions of n. The dissections allow us to prove a conjecture of Hirschhorn concerning the 2^n-dissection of $(q;q)_\infty$, as well as determine the pattern of the sign changes of the coefficients a_n of the infinite product

$$\frac{(q^{2^{k-1}}; q^{2^{k-1}})_\infty}{(q^p;q^p)^2_\infty} = \sum_{n=0}^{\infty} a_n q^n, \quad k \geq 1, p \geq 5 \text{ a prime}.$$

This covers a recent result of Bringmann et al. that corresponds to the case $k=1$ and $p=5$.

1. Introduction

In a recent paper [6], the authors made the following observation. Let S denote the set of positive integers that are not multiples of 3, and let $D_S(n)$ denote the number of partitions of n into an even number of distinct parts from S minus the number of partitions of n into an odd number of distinct parts from S. From the known 3-dissection for $(q;q)_\infty$, it follows that

$$(1.1)\quad \sum_{n=0}^{\infty} D_S(n) q^n = (q, q^2; q^3)_\infty =$$

$$\frac{1}{(q^3, q^6, q^9, q^{18}, q^{21}, q^{24}; q^{27})_\infty} - \frac{q}{(q^3, q^9, q^{12}, q^{15}, q^{18}, q^{24}; q^{27})_\infty} - \frac{q^2}{(q^6, q^9, q^{12}, q^{15}, q^{18}, q^{21}; q^{27})_\infty},$$

where hereafter we adopt the standard notation

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad (a_1,\ldots,a_j;q)_\infty := (a_1;q)_\infty \cdots (a_j;q)_\infty.$$

This dissection implies $D_S(3n) \to \infty$ and $D_S(3n+1)$, $D_S(3n+2) \to -\infty$ as $n \to \infty$. The behavior is very different from what happens if we replace S with $\mathbb{N}$ and let $D_{\mathbb{N}}(n)$ be the number of partitions

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of n into an even number of distinct parts from $\mathbb{N}$ minus the number of partitions of n into an odd number of distinct parts from $\mathbb{N}$. Franklin's proof of the pentagonal number theorem showed

$$\{D_{\mathbb{N}}(n) \mid n \in \mathbb{N}\} = \{-1, 0, 1\}.$$

Both sides of (1.1) may be interpreted in terms of restricted partitions. For $a \in \{1, 2, 3\}$, let $p_{a,9}(n)$ denote the number of partitions of n into parts $\not\equiv \pm a, 0 \pmod 9$. Then

$$\begin{aligned} D_S(3n) &= p_{4,9}(n), \\ D_S(3n+1) &= -p_{2,9}(n), \\ D_S(3n+2) &= -p_{1,9}(n). \end{aligned} \tag{1.2}$$

A natural question is to ask if identities like (1.2) hold more widely. To investigate this problem, we first recall that the identity (1.1) came from the 3-dissection of the infinite product $(q;q)_\infty$ and note that $(q;q)_\infty$ is just the specialization of the *quintuple product*

$$Q(z,q) := \frac{\langle z^2; q\rangle_\infty}{(-z, -q/z; q)_\infty}, \tag{1.3}$$

to the case of $q \to q^4$ and $z \to q$, where $\langle a;q\rangle_\infty := (a, q/a, q; q)_\infty$, a quantity that for ease of notation we will refer to as a *(Jacobi) triple product*, noting also, for later use, that $\langle a;q\rangle_\infty = \langle q/a; q\rangle_\infty$. This indicates that identities analogous to those at (1.2) may be systematically produced via the m-dissection of quintuple products of the form $Q(q^j, q^M)$ and prompts us to consider the general m-dissection of the general quintuple product $Q(z,q)$. Following this leads us to discover and prove Theorem 1.1.

Theorem 1.1. *Let $|q| < 1$ and $z \neq 0$ and m a positive integer such that $3 \nmid m$.*

(i) If $m \equiv 1 \pmod 3$, then

$$Q(z,q) = \sum_{r=0}^{m-1} q^{\frac{1}{2}r(3r-1)} z^{3r} Q\left(z^m q^{\frac{1}{6}m(m+6r-1)}, q^{m^2}\right). \tag{1.4}$$

(ii) If $m \equiv 2 \pmod 3$, then

$$Q(z,q) = \sum_{r=0}^{m-1} q^{\frac{1}{2}r(3r-1)} z^{3r} Q\left(z^{-m} q^{\frac{1}{6}m(m-6r+1)}, q^{m^2}\right). \tag{1.5}$$

Upon making the substitutions $q \to q^M$, $z \to q^j$ one readily derives m-dissections for the quintuple product $Q(q^j, q^M) = \left(q^j, q^{M-j}, q^M; q^M\right)_\infty \left(q^{M-2j}, q^{M+2j}; q^{2M}\right)_\infty$. In particular, as is noted before, specializing (j, M) to $(1, 4)$ one has that $Q(q, q^4) = (q, q^3, q^4; q^4)_\infty (q^2, q^6; q^8)_\infty = (q;q)_\infty$. Applying the m-dissection identities (1.4) and (1.5) to this case, we are able to address the question concerning relations between partition functions that was raised at the beginning and initiated this work. We deduce other general partition identities, of which the following is a representative example (recall that $b_m(n)$ denotes the number of m-regular partitions of n).

Theorem 1.2. *Let $m \geq 5$ be an integer relatively prime to 6 and square-free, and let S be the set of positive integers containing no multiples of m. Define $D_S(n)$ to be the number of partitions of n into an even number of distinct parts from S minus the number of partitions of n into an odd number of distinct parts from S.*

(i) Define $r = (m^2-1)/24$. If $m \equiv 1 \pmod 6$ set $s = (m-1)/6$, and if $m \equiv -1 \pmod 6$ set $s = (m+1)/6$. Then

$$D_S(mn + r) = (-1)^s b_m(n), \qquad \text{for all } n \geq 0. \tag{1.6}$$

(ii) If $m \equiv 1 \pmod 6$*, define*

$$(1.7)\quad T := \left\{ \frac{u(3u-1)}{2} \middle| 0 \le u \le \frac{m-1}{3} \right\} \cup \left\{ \frac{u(3u+1)}{2} \middle| 1 \le u \le \frac{m-7}{6} \right\} \cup \left\{ \frac{m^2-1}{24} \right\} \pmod m,$$

and if $m \equiv -1 \pmod 6$*, define*

$$(1.8)\quad T := \left\{ \frac{u(3u-1)}{2} \middle| 0 \le u \le \frac{m-5}{6} \right\} \cup \left\{ \frac{u(3u+1)}{2} \middle| 1 \le u \le \frac{m-2}{3} \right\} \cup \left\{ \frac{m^2-1}{24} \right\} \pmod m.$$

If $v \in \{0, 1, 2, \dots, m-1\} \setminus T$*, then* $D_S(mn+v) = 0$*, for all* $n \ge 0$*.*

Remark 1. In the present paper we do not consider combinatorial proofs of the various partition identities stated. However, we state here a possible approach to finding combinatorial proofs of all of the identities that involve a function of the form $D_S(n)$ for some set of positive integers S. Taking (1.6) as an example, this approach involves finding an injection from the set of partitions of $mn+r$ into an even number of distinct parts from S to the set of partitions of $mn+r$ into an odd number of distinct parts from S (or vice versa), and then showing that there is a bijection between the set of partitions not matched up in this way and the partitions counted by $b_m(n)$. However, we caution that this approach may not be feasible and that other approaches may be advantageous.

As an example of a partition theoretic result derived from the m-dissection of quintuple products we give the following example. See Subsection 3.2 for part (ii) of the example, and recall that $P_{a,M}(n)$ counts the number of partitions of n into parts $\not\equiv \pm a, 0 \pmod M$, $\not\equiv M \pm 2a \pmod{2M}$.

Example 3. *Let* $P_{a,M}(n)$ *be defined as at* (3.2).
(i) Let S *be the set of positive integers* $\equiv \pm 1, \pm 3, \pm 4 \pmod{10}$*. Then*

$$\begin{aligned}
D_S(5n) &= P_{4,25}(n)\\
D_S(5n+1) &= -P_{6,25}(n)\\
D_S(5n+2) &= P_{9,25}(n-1)\\
D_S(5n+3) &= -P_{1,25}(n)\\
D_S(5n+4) &= -P_{11,25}(n-2)
\end{aligned}$$

Besides its role in the study of partition identities, another interesting application arising from Theorem 1.1 is to prove Hirschhorn's conjecture [5] about the 2^n-dissection of $(q;q)_\infty = Q(q, q^4)$. We depart slightly from the notation used by Hirschhorn.

Theorem 1.3 (Hirschhorn's conjecture)**.** *Let* $n \ge 1$ *be an integer and let* $m = 2^n$*. Then the* m*-dissection of* $(q;q)_\infty$ *is give by*

$$(1.9)\quad (q;q)_\infty = \sum_{k=1}^{m} (-1)^{k+\epsilon} q^{c_k} (q^{2(2k-1)m}, q^{8m^2-2(2k-1)m}; q^{8m^2})_\infty (q^{2m^2-(2k-1)m}, q^{2m^2+(2k-1)m}, q^{4m^2}; q^{4m^2})_\infty,$$

where $\epsilon = 0$ *(respectively, 1) if* n *is odd (respectively, even), and for* $k = 1, 2, 3, \dots, 2^n$*,*

$$(1.10)\quad c_k = \begin{cases} P\left(\frac{2m-1}{3} - (k-1)\right), & \text{if } n \text{ is odd,}\\[2ex] P\left(-\frac{2m-2}{3} + (k-1)\right), & \text{if } n \text{ is even,}\end{cases}$$

where $P(t) = t(3t-1)/2$*.*

Example 5 includes partition identities from this dissection.

Finally, in Section 5, as another relevant implication of Theorem 1.1, we give a number of results about periodicity of signs of coefficients in the series expansion of various infinite products, of which the following is a prime example that covers a recent result [2, Theorem 1.1] of Bringmann et al..

Theorem 1.4. *Let $p > 3$ be a prime. For $k \geq 1$, write*

$$\frac{(q^{2^{k-1}}; q^{2^{k-1}})_\infty}{(q^p; q^p)_\infty^2} = \sum_{n=0}^{\infty} a_n q^n.$$

Then

(1) if $p \equiv 1 \pmod 3$, one has that

$$a_n \begin{cases} > 0 & \text{if } n \equiv 3 \cdot 2^k r^2 + 2^{k-1} r \pmod p \text{ with } 0 \leq r < \frac{4(2p+1)-6}{24} \text{ or } \frac{4(5p+1)-6}{24} < r \leq p-1, \\ & \qquad \text{and } n \geq \mathcal{L}(r,k), \\ = 0 & \text{if } n \not\equiv 3 \cdot 2^k r^2 + 2^{k-1} r \pmod p, \\ < 0 & \text{if } n \equiv 3 \cdot 2^k r^2 + 2^{k-1} r \pmod p \text{ with } \frac{4(2p+1)-6}{24} < r < \frac{4(5p+1)-6}{24}, \\ & \qquad \text{and } n \geq \mathcal{L}(r,k), \end{cases}$$

(2) if $p \equiv -1 \pmod 3$, one has that

$$a_n \begin{cases} > 0 & \text{if } n \equiv 3 \cdot 2^k r^2 + 2^{k-1} r \pmod p \text{ with } 0 \leq r < \frac{4(p+1)-6}{24} \text{ or } \frac{4(4p+1)-6}{24} < r \leq p-1, \\ & \qquad \text{and } n \geq \mathcal{L}(r,k), \\ = 0 & \text{if } n \not\equiv 3 \cdot 2^k r^2 + 2^{k-1} r \pmod p, \\ < 0 & \text{if } n \equiv 3 \cdot 2^k r^2 + 2^{k-1} r \pmod p \text{ with } \frac{4(p+1)-6}{24} < r < \frac{4(4p+1)-6}{24}, \\ & \qquad \text{and } n \geq \mathcal{L}(r,k), \end{cases}$$

where $\mathcal{L}(r,k)$ is defined by

$$t_1 = t_1(r) = 2^{k+1} p(p + (6r-1))/6 + p \pmod{2^{k+1} p^2},$$

$$t_2 = t_2(r) = 2^{k+1} p^2 + 2^k p + 2^{k+1}(p + (6r-1))p/3 \pmod{2^{k+2} p^2},$$

$$\mathcal{L}(r,k) = \frac{7 \cdot 2^{k-1} p^2}{6} + \frac{1}{2} t_1 \left(\frac{t_1}{2^{k+1} p^2} - 1 \right) + \frac{1}{2} t_2 \left(\frac{t_2}{2^{k+2} p^2} - 1 \right) - \frac{2^k}{48}.$$

In particular, when $k = 1$ and $p = 5$, the resulting conclusion yields

Corollary 1.1 (Bringmann–Han–Heim–Kane)**.** *For*

$$\frac{(q;q)_\infty}{(q^5;q^5)_\infty^2} = \sum_{n=0}^{\infty} c_n q^n,$$

one has that

$$c_n \begin{cases} > 0 & \text{if } n \equiv 0 \pmod 5, \\ = 0 & \text{if } n \equiv 3, 4 \pmod 5, \\ < 0 & \text{if } n \equiv 1, 2 \pmod 5. \end{cases}$$

The remainder of this work is organized as follows. In Section 2, we first review a number of classical identities related to triple products that will be useful for proving Theorem 1.1. After these, we give the proof of Theorem 1.1 as well as multiple useful implications of the theorem. In Section 3 we apply the m-dissection of $Q(z,q)$ to prove Theorem 1.2 and discuss various novel examples of partition identities following from it. The last two sections primarily discuss other applications of Theorem 1.1 and its implications; more specifically, in Section 4 we prove a conjecture of Hirschhorn, namely, Theorem 1.3, from which we determine the sign of the coefficients in the series expansion of $(q;q)_\infty/(q^m;q^m)_\infty$ for $m=2^n$. In Section 5 we give a proof of Theorem 1.4 and study the signs of the coefficients coming from other infinite product classes.

2. m-Dissection of the Quintuple Product $Q(z,q)$

We recall two equivalent versions of the Jacobi triple product identity:

Theorem 2.1. *For $|q|<1$ and $z\neq 0$,*

$$\sum_{n=-\infty}^{\infty}(-z)^n q^{n^2}=\langle zq;q^2\rangle_\infty, \tag{2.1}$$

and

$$\sum_{n=-\infty}^{\infty}(-z)^n q^{n(n-1)/2}=\langle z;q\rangle_\infty. \tag{2.2}$$

If the bilateral series in either (2.1) or (2.2) are split in to m series, each containing terms corresponding to one of the m arithmetic progressions

$$\{mk+r|k\in\mathbb{Z}\},\qquad r=0,1,2,\dots,m-1,$$

and then either (2.1) or (2.2) is used again to sum each of these m bilateral series, one arrives at the following m-dissections.

Corollary 2.1. *If $|q|<1$, $z\neq 0$ and m is a positive integer, then*

$$\langle zq;q^2\rangle_\infty=\sum_{r=0}^{m-1}(-z)^r q^{r^2}\left\langle(-1)^{m+1}z^m q^{m^2+2mr};q^{2m^2}\right\rangle_\infty, \tag{2.3}$$

and

$$\langle z;q\rangle_\infty=\sum_{r=0}^{m-1}(-z)^r q^{(r^2-r)/2}\left\langle(-1)^{m+1}z^m q^{(m^2-m)/2+mr};q^{m^2}\right\rangle_\infty. \tag{2.4}$$

We will sometimes use the following elementary results (for integers r and k, where $1\leq r<k$)

$$(aq^{-r};q^k)_\infty=(1-aq^{-r})(aq^{k-r};q^k)_\infty,\qquad (aq^{k+r};q^k)_\infty=\frac{(aq^r;q^k)_\infty}{1-aq^r}. \tag{2.5}$$

We also make use of the quintuple product identity (see [3] for an excellent survey article on the quintuple product identity and its many proofs).

Theorem 2.2. *For $|q|<1$ and $z\neq 0$,*

$$\begin{aligned}\sum_{n=-\infty}^{\infty}q^{n(3n-1)/2}z^{3n}(1-zq^n)&=\langle -qz^3;q^3\rangle_\infty - z\langle -q^2z^3;q^3\rangle_\infty\\ &=\langle z;q\rangle_\infty(qz^2,q/z^2;q^2)_\infty=\frac{\langle z^2;q\rangle_\infty}{(-z,-q/z;q)_\infty}.\end{aligned} \tag{2.6}$$

In a previous paper [8, Theorem 3.1], the second author gave the m-dissection of $Q(q^t,q^m)$ in terms of other quintuple products, in the case where $\gcd(m,6)=1$ and t is an integer such that $1\leq t<m/2$ and $\gcd(t,m)=1$. With $Q(z,q)$ as at (1.3), define

$$\mathcal{Q}(t,m):=Q(q^t,q^m). \tag{2.7}$$

Theorem 2.3. *Let $m \geq 5$ be an integer relatively prime to 6 and let $t \in \{1, 2, \dots, (m-1)/2\}$ such that* $\gcd(t, m) = 1$. *Let $\mathcal{Q}(t, m)$ be as defined at (2.7).*
(i)If $m \equiv 1 \pmod 6$, then

$$\begin{aligned}
(2.8)\quad \mathcal{Q}(t,m) = &\sum_{r=0}^{\frac{m-1}{3}} (-1)^r q^{\frac{1}{2}r(m(3r-1)+6t)} \mathcal{Q}\left(\frac{1}{6}m\left(m^2+m(6r-1)+6t\right), m^3\right) \\
&+ \sum_{r=0}^{\frac{m-4}{3}} (-1)^r q^{\frac{1}{6}(m-3r-2)\left(m^2-m(3r+1)-6t\right)} \mathcal{Q}\left(\frac{1}{2}m\left(m^2-m(2r+1)-2t\right), m^3\right) \\
&+ \sum_{r=0}^{\frac{m-7}{6}} (-1)^r q^{\frac{1}{2}(3r+1)(mr+2t)} \mathcal{Q}\left(\frac{1}{6}m(m(m-6r-1)-6t), m^3\right) \\
&+ \sum_{r=0}^{\frac{m-7}{6}} (-1)^{\frac{1}{6}(m+6r+11)} q^{\frac{1}{24}(m-6r-1)\left(m^2-6mr+m-12t\right)} \mathcal{Q}\left(m(mr+t), m^3\right).
\end{aligned}$$

(ii) If $m \equiv 5 \pmod 6$, then

$$\begin{aligned}
(2.9)\quad \mathcal{Q}(t,m) = &\sum_{r=0}^{\frac{m-5}{6}} (-1)^r q^{\frac{1}{2}r(m(3r-1)+6t)} \mathcal{Q}\left(\frac{1}{6}m\left(m^2-6mr+m-6t\right), m^3\right) \\
&+ \sum_{r=0}^{\frac{m-5}{6}} (-1)^{\frac{1}{6}(m+6r+1)} q^{\frac{1}{24}(m-6r-1)\left(m^2-6mr+m-12t\right)} \mathcal{Q}\left(m(mr+t), m^3\right) \\
&+ \sum_{r=0}^{\frac{m-2}{3}} (-1)^r q^{\frac{1}{2}(3r+1)(mr+2t)} \mathcal{Q}\left(\frac{1}{6}m\left(m^2+6mr+m+6t\right), m^3\right) + \\
&\sum_{r=0}^{\frac{m-5}{3}} (-1)^{r+1} q^{\frac{1}{6}(m-3r-2)(m(m-3r-1)-6t)} \mathcal{Q}\left(\frac{1}{2}m(m(m-2r-1)-2t), m^3\right).
\end{aligned}$$

It was believed by the second author at the time of writing [8] that it was not possible to give the m-dissection of the general quintuple product $Q(z, q)$. However, while further investigating the type of results exhibited at (1.2), it was discovered that such an m-dissection was possible for integers $m > 1$ such that $\gcd(3, m) = 1$.

The idea behind our plan to find m-dissections for $Q(z, q)$ as at (1.3) is to first of all use (2.6) to write it as a combination of triple products,

$$Q(z,q) = \langle -qz^3; q^3 \rangle_\infty - z\langle -q^2 z^3; q^3 \rangle_\infty \tag{2.10}$$

then use one of the m-dissections in Corollary 2.1 to get the m-dissections of each of these triple products, and then find ways of matching up pairs of triple products from these m-dissections in such a way as to use (2.6) once again to combine them into quintuple products, giving the desired results. A couple of comments are in order before starting the proof.

- The m-dissections are only for m of the form $6t - 1$ or $6t + 1$, t a positive integer.
- For m of the form $6t + 2$ or $6t + 4$ we get $m/2$-dissections.
- For $m \equiv 0 \pmod 3$, the m-dissection of $Q(z, q)$ does not appear to have a straightforward description, at least not for all such m.

We are now in a position to prove Theorem 1.1:

Proof of Theorem 1.1. After applying (2.4) to the two triple products on the right side of (2.10) (with q replaced with q^3 and z replaced, respectively, with $-qz^3$ and $-q^2z^3$, and then simplifying, one gets

$$(2.11)\quad Q(z,q)=\sum_{r=0}^{m-1} q^{r(3r-1)/2}z^{3r}\left\langle -q^{m(3m+6r-1)/2}z^{3m};q^{3m^2}\right\rangle_\infty \\ -z\sum_{r=0}^{m-1} q^{r(3r+1)/2}z^{3r}\left\langle -q^{m(3m+6r+1)/2}z^{3m};q^{3m^2}\right\rangle_\infty =: \sum_{r=0}^{m-1} a_r - z\sum_{r=0}^{m-1} b_r.$$

One can clearly see at this point that if m is even, then we get an $m/2$ dissection of $Q(z,q)$. How we proceed next depends on whether $m\equiv 1 \pmod 3$ or $m\equiv 2 \pmod 3$. Also, for use below, it is a straightforward, if tedious, task to showing, using (2.5) above, that if r is replaced with $r+m$ or $r-m$ in b_r, then b_r is unchanged, so that below when we mention a term b_s, where $s=r+m$, we mean b_r.

(i) We now move to $m\equiv 1 \pmod 3$. If we form the combination $a_{r_1}-zb_{r_2}$, we get

$$a_{r_1}-zb_{r_2} = q^{\frac{r_1(3r_1-1)}{2}}z^{3r_1}\left\langle -q^{\frac{m(3m+6r_1-1)}{2}}z^{3m};q^{3m^2}\right\rangle - zq^{\frac{r_2(3r_2+1)}{2}}z^{3r_2}\left\langle -q^{\frac{m(3m+6r_2+1)}{2}}z^{3m};q^{3m^2}\right\rangle \\ = q^{\frac{r_1(3r_1-1)}{2}}z^{3r_1}\left(\left\langle -q^{m^2}\left(q^{\frac{m(m+6r_1-1)}{6}}z^m\right)^3;q^{3m^2}\right\rangle \right. \\ \left. - zq^{\frac{r_2(3r_2+1)}{2}-\frac{r_1(3r_1-1)}{2}}z^{3(r_2-r_1)}\left\langle -q^{2m^2}\left(q^{\frac{m(-m+6r_2+1)}{6}}z^m\right)^3;q^{3m^2}\right\rangle\right).$$

To turn this into a quintuple product Q, one must have

$$q^{\frac{m(m+6r_1-1)}{6}}z^m = q^{\frac{m(-m+6r_2+1)}{6}}z^m = zq^{\frac{r_2(3r_2+1)}{2}-\frac{r_1(3r_1-1)}{2}}z^{3(r_2-r_1)},$$

and this forces r_2 and r_1 to satisfy that

$$r_2 = r_1+\frac{m-1}{3}.$$

After the replacement $r\to r+(m-1)/3$ in b_r, one gets (after simplifying and recalling the remark above about b_r being unchanged if r is replaced with $r+m$ if $r+(m-1)/3>m$) that

$$(2.12)\quad a_r - zb_{r+(m-1)/3} = q^{r(3r-1)/2}z^{3r} \\ \times\left[\left\langle -\left(q^{m(m+6r-1)/6}z^m\right)^3 q^{m^2};q^{3m^2}\right\rangle_\infty - q^{m(m+6r-1)/6}z^m\left\langle -\left(q^{m(m+6r-1)/6}z^m\right)^3 q^{2m^2};q^{3m^2}\right\rangle_\infty\right] \\ = q^{r(3r-1)/2}z^{3r}Q(q^{m(m+6r-1)/6}z^m,q^{m^2}),$$

where the last equality follows from (2.10) (with the replacements $z\to q^{m(m+6r-1)/6}z^m$ and $q\to q^{m^2}$). Upon summing (2.12) over $0\le r\le m-1$ completes the proof of (i).

(ii) For $m\equiv 2 \pmod 3$, clearly the replacement $r\to r+(m-1)/3$ does not make sense, so instead we recall that $\langle y;q\rangle_\infty = \langle q/y;q\rangle_\infty$ and rewrite $a_{r_1}-zb_{r_2}$ as

$$a_{r_1}-zb_{r_2} = q^{\frac{r_1(3r_1-1)}{2}}z^{3r_1}\left\langle -q^{\frac{m(3m-6r_1+1)}{2}}z^{-3m};q^{3m^2}\right\rangle - zq^{\frac{r_2(3r_2+1)}{2}}z^{3r_2}\left\langle -q^{\frac{m(3m-6r_2-1)}{2}}z^{-3m};q^{3m^2}\right\rangle \\ = q^{\frac{r_1(3r_1-1)}{2}}z^{3r_1}\left(\left\langle -q^{m^2}\left(q^{\frac{m(m-6r_1+1)}{6}}z^{-m}\right)^3;q^{3m^2}\right\rangle \right. \\ \left. - zq^{\frac{r_2(3r_2+1)}{2}-\frac{r_1(3r_1-1)}{2}}z^{3(r_2-r_1)}\left\langle -q^{2m^2}\left(q^{\frac{m(-m-6r_2-1)}{6}}z^{-m}\right)^3;q^{3m^2}\right\rangle\right).$$

This time, to convert the term inside parentheses into a quintuple product Q, one must have

$$q^{\frac{m(m-6r_1+1)}{6}}z^{-m}=q^{\frac{m(-m-6r_2-1)}{6}}z^{-m}=zq^{\frac{r_2(3r_2+1)}{2}-\frac{r_1(3r_1-1)}{2}}z^{3(r_2-r_1)},$$

and this forces r_2 and r_1 to satisfy that

$$r_2=r_1-\frac{m+1}{3}.$$

We proceed as above and make the replacement $r\to r-(m+1)/3$ in b_r, this time recalling that b_r is unchanged if r is replaced with $r-m$ in b_r, to get that

$$(2.13)\quad a_r-zb_{r-(m+1)/3}=q^{r(3r-1)/2}z^{3r}\times$$
$$\left[\left\langle -q^{m^2}\left(q^{m(m-6r+1)/6}z^{-m}\right)^3;q^{3m^2}\right\rangle_\infty - q^{m(m-6r+1)/6}z^{-m}\left\langle -q^{2m^2}\left(q^{m(m-6r+1)/2}z^{-m}\right)^3;q^{3m^2}\right\rangle_\infty\right]$$
$$=q^{r(3r-1)/2}z^{3r}Q(q^{m(m-6r+1)/6}z^{-m},q^{m^2}),$$

where the last equality again follows from (2.10) (this time with the replacements $q\to q^{m^2}$ and $z\to q^{m(m-6r+1)/6}z^{-m}$). As above, summing (2.13) over $0\le r\le m-1$ completes the proof of (ii). □

Of course the parameters in the above theorem can be specialized to give a version of the theorem in the earlier paper for the m-dissection of $Q(q^t,q^m)$. We say a "version" of that previous theorem, because the statement of the previous theorem was more complicated. Namely, the sums corresponding to $\sum_{r=0}^{m-1}$ in parts (i) and (ii) above were split into four sums in the previous theorem, so as to avoid quintuple products in the m-dissections which had powers of q with negative exponents. Simply specializing z and q in parts (i) and (ii) above almost certainly will lead to m-dissections in which some of the resulting quintuple products will have powers of q with negative exponents. If only positive exponents are required, then (2.5) may be employed to remove them.

However, Theorem 1.1 has two major advantages over the previous theorem. Firstly, it can be employed to derive m-dissections when m is of the form $m=6t+2$ or of the form $m=6t+4$, whereas the previous theorem could not provide those. Secondly, the previous theorem could only provide m-dissections of quintuple products of the form $Q(q^t,q^m)$, whereas Theorem 1.1 can provide m-dissections of quintuple products of the form $Q(q^t,q^k)$, where k is independent of m.

While the initial impetus that led to the discovery of Theorem 1.1 was further investigation of the phenomenon exhibited at (1.2), we provide other applications in later sections.

For later use we state the expansions derived making the replacements $q\to q^M$ and $z\to q^j$ in (1.4) and (1.5), where $M>3$ is an integer and j is a positive integer satisfying $1\le j<M/2$. We also use the product form of $Q(z,q)$ (rather than the fractional form):

$$Q(z,q)=(z,q/z,q;q)_\infty(qz^2,q/z^2;q^2)_\infty.$$

With these substitutions we arrive at the following corollary.

Corollary 2.2. *Let* $|q|<1$ *and let* $M>3$ *be an integer and let* j *is a positive integer satisfying* $1\le j<p/2$. *Let* m *be a positive integer such that* $3\nmid m$.

(i) If $m\equiv 1 \pmod 3$*, then*

$$(2.14)\quad \left(q^j,q^{M-j},q^M;q^M\right)_\infty\left(q^{M-2j},q^{M+2j};q^{2M}\right)_\infty$$
$$=\sum_{r=0}^{m-1}q^{M(3r-1)r/2+3jr}\left(q^{mM(m+6r-1)/6+jm},q^{m^2M-mM(m+6r-1)/6-jm},q^{m^2M};q^{m^2M}\right)_\infty$$
$$\times\left(q^{m^2M+2jm+M(m+6r-1)m/3},q^{m^2M-2jm-mM(m+6r-1)/3};q^{2m^2M}\right)_\infty.$$

(ii) If $m \equiv 2 \pmod 3$*, then*

$$(2.15)\quad \left(q^j, q^{M-j}, q^M; q^M\right)_\infty \left(q^{M-2j}, q^{M+2j}; q^{2M}\right)_\infty$$
$$= \sum_{r=0}^{m-1} q^{M(3r-1)r/2+3jr} \left(q^{mM(m-6r+1)/6-jm}, q^{m^2M-mM(m-6r+1)/6+jm}, q^{m^2M}; q^{m^2M}\right)_\infty$$
$$\times \left(q^{m^2M+2jm-M(m-6r+1)m/3}, q^{m^2M-2jm+mM(m-6r+1)/3}; q^{2m^2M}\right)_\infty.$$

It can be seen that all of the q-products in (2.14) and (2.15) are functions of q^m, so that the sums on the right sides of (2.14) and (2.15) give m-dissection of the q-product on the left-side. Some components in the m-dissection may be formed by combining two or more terms in these sums.

The next two lemmas are needed as preparation for the proof of Theorem 2.4, which gives a general formula for rewriting the m-dissection of $Q(q^j, q^M)$ stated in Corollary 2.2 in such a way that any negative exponents of q in the expansion are removed.

Lemma 2.1 ([7])**.** *For* $z = (Q_1, Q_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ *and* $q_z = e^{2\pi i(Q_1\tau+Q_2)}$*, define*

$$K_{(Q_1,Q_2)}(\tau) = e^{\pi i Q_2(Q_1-1)} q^{\frac{1}{2}Q_1(Q_1-1)}(1-q_z)\prod_{n=1}^{\infty}(1-q_zq^n)(1-q_z^{-1}q^n)(1-q^n)^{-2}$$

called a Klein form. Then for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$*,*

$$K_{(Q_1,Q_2)}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{-1} K_{(Q_1a+Q_2c, Q_1b+Q_2d)}(\tau).$$

Lemma 2.2. *Let* $q = \exp(2\pi i\tau)$ *with* $\tau \in \mathbb{H}$*. Then for any integer* $M \geq 2$ *and* $1 \leq j < M/2$*, the infinite product*

$$q^\ell(q^j, q^{M-j}, q^M; q^M)_\infty (q^{M-2j}, q^{M+2j}; q^{2M})_\infty$$

is modular if and only if

$$\ell = \frac{7M}{24} + \frac{1}{2}j\left(\frac{j}{M}-1\right) + \frac{1}{2}(M-2j)\left(\frac{M-2j}{2M}-1\right).$$

Proof. Following the definition of the Klein form, it is routine to verify that

$$q^\ell(q^j, q^{M-j}, q^M; q^M)_\infty (q^{M-2j}, q^{M+2j}; q^{2M})_\infty$$
$$= K_{(j/M,0)}(M\tau) K_{((M-2j)/2M,0)}(2M\tau)\eta(M\tau)^3\eta(2M\tau)^2,$$

and the conclusion follows by Lemma 2.1. □

Theorem 2.4. *For* $M \geq 3$*,* $1 \leq j < M/2$*, and* $m \equiv \pm 1 \pmod 3$*, one has that*

$$(2.16)\quad (q^j, q^{M-j}, q^M; q^M)_\infty (q^{M-2j}, q^{M+2j}; q^{2M})_\infty$$
$$= \sum_{r=0}^{m-1} (-1)^{s(r)} q^{\mathcal{L}(r)} (q^{t_1(r)}, q^{m^2M-t_1(r)}, q^{m^2M}; q^{m^2M})_\infty (q^{t_2(r)}, q^{2m^2M-t_2(r)}; q^{2m^2M})_\infty,$$

where

$$t_1(r) = mM(m \pm (6r-1))/6 \pm jm \pmod{m^2M},$$
$$t_2(r) = m^2M + 2jm \pm M(m \pm (6r-1))m/3 \pmod{2m^2M},$$
$$\mathcal{L}(r) = \frac{7m^2M}{24} + \frac{1}{2}t_1(r)\left(\frac{t_1(r)}{m^2M}-1\right) + \frac{1}{2}t_2(r)\left(\frac{t_2(r)}{2m^2M}-1\right)$$

$$-\left(\frac{7M}{24}+\frac{1}{2}j\left(\frac{j}{M}-1\right)+\frac{1}{2}(M-2j)\left(\frac{M-2j}{2M}-1\right)\right),$$

and if $m\equiv 1 \pmod 3$*, then*

$$s(r)=\begin{cases}0, & \textit{if } r\le \frac{(2m+1)M-6j}{6M},\\ 1, & \textit{if } \frac{(2m+1)M-6j}{6M}<r\le \frac{(5m+1)M-6j}{6M},\\ 2, & \textit{if } r>\frac{(5m+1)M-6j}{6M},\end{cases}\tag{2.17}$$

and if $m\equiv -1 \pmod 3$*, then*

$$s(r)=\begin{cases}0, & \textit{if } r\le \frac{(m+1)M-6j}{6M},\\ 1, & \textit{if } \frac{(m+1)M-6j}{6M}<r\le \frac{(4m+1)M-6j}{6M},\\ 2, & \textit{if } r>\frac{(4m+1)M-6j}{6M}.\end{cases}\tag{2.18}$$

Proof. By Lemma 2.2 for $m\equiv \pm 1 \pmod 3$, we have that for $q=\exp(2\pi i\tau)$,

$$\begin{aligned}&q^{\ell}\left(q^j,q^{M-j},q^M;q^M\right)_\infty\left(q^{M-2j},q^{M+2j};q^{2M}\right)_\infty\\ &=q^{\ell}\sum_{r=0}^{m-1}q^{M(3r-1)r/2+3jr}\left(q^{mM(m\pm(6r-1))/6\pm jm},q^{m^2M-mM(m\pm(6r-1))/6\mp jm},q^{m^2M};q^{m^2M}\right)_\infty\\ &\quad\times\left(q^{m^2M+2jm\pm p(m\pm(6r-1))m/3},q^{m^2M-2jm\mp mM(m\pm(6r-1))/3};q^{2m^2M}\right)_\infty,\end{aligned}$$

where

$$\ell=\frac{7M}{24}+\frac{1}{2}j\left(\frac{j}{M}-1\right)+\frac{1}{2}(M-2j)\left(\frac{M-2j}{2M}-1\right),$$

is modular. Notice that each of the components

$$\begin{aligned}&\left(q^{mM(m\pm(6r-1))/6\pm jm},q^{m^2M-mM(m\pm(6r-1))/6\mp jm},q^{m^2M};q^{m^2M}\right)_\infty\\ &\quad\times\left(q^{m^2M+2jm\pm M(m\pm(6r-1))m/3},q^{m^2M-2jm\mp mM(m\pm(6r-1))/3};q^{2m^2M}\right)_\infty\end{aligned}$$

is uniquely determined by an $r=0,\dots,m-1$ as a part of the m-dissection of the infinite product $\left(q^j,q^{M-j},q^M;q^M\right)_\infty\left(q^{M-2j},q^{M+2j};q^{2M}\right)_\infty$ corresponding to some residue class, and recall that components of the m-dissection of a modular form remain modular. Also notice that

$$\begin{aligned}|mM(m\pm(6r-1))/6\pm jm|&<2m^2M,\\ |m^2M+2jm\pm M(m\pm(6r-1))m/3|&<4m^2M,\end{aligned}$$

so these imply that the product

$$\begin{aligned}&\left(q^{mM(m\pm(6r-1))/6\pm jm},q^{m^2M-mM(m\pm(6r-1))/6\mp jm},q^{m^2M};q^{m^2M}\right)_\infty\\ &\quad\times\left(q^{m^2M+2jm\pm M(m\pm(6r-1))m/3},q^{m^2M-2jm\mp mM(m\pm(6r-1))/3};q^{2m^2M}\right)_\infty\end{aligned}$$

differs from the product

$$(q^{t_1},q^{m^2M-t_1},q^{m^2M};q^{m^2M})_\infty(q^{t_2},q^{2m^2M-t_2};q^{2m^2M})_\infty$$

by some power of q, where $t_1=t_1(r)$ and $t_2=t_2(r)$. Thus, by Lemma 2.2 one can tell that

$$\begin{aligned}&q^{\ell+M(3r-1)r/2+3jr}\left(q^{mM(m\pm(6r-1))/6\pm jm},q^{m^2M-mM(m\pm(6r-1))/6\mp jm},q^{m^2M};q^{m^2M}\right)_\infty\\ &\quad\times\left(q^{m^2M+2jm\pm M(m\pm(6r-1))m/3},q^{m^2M-2jm\mp mM(m\pm(6r-1))/3};q^{2m^2M}\right)_\infty\end{aligned}$$

must be modular. Then the modularity of the infinite product and Lemma 2.2 once again force it to be of the form

$$(-1)^s q^L (q^{t_1}, q^{m^2M-t_1}, q^{m^2M}; q^{m^2M})_\infty (q^{t_2}, q^{2m^2M-t_2}; q^{2m^2M})_\infty$$

with

$$L = \frac{7m^2M}{24} + \frac{1}{2}t_1\left(\frac{t_1}{m^2M} - 1\right) + \frac{1}{2}t_2\left(\frac{t_2}{2m^2M} - 1\right)$$

for some $s = s(r)$, whence,

$$\begin{aligned}&(q^j, q^{M-j}, q^M; q^M)_\infty (q^{M-2j}, q^{M+2j}; q^{2M})_\infty \\ &= \sum_{r=0}^{m-1} (-1)^{s(r)} q^{\mathcal{L}(r)} (q^{t_1(r)}, q^{m^2M-t_1(r)}, q^{m^2M}; q^{m^2M})_\infty (q^{t_2(r)}, q^{2m^2M-t_2(r)}; q^{2m^2M})_\infty\end{aligned}$$

with

$$\begin{aligned}\mathcal{L}(r) = &\frac{7m^2M}{24} + \frac{1}{2}t_1\left(\frac{t_1}{m^2M} - 1\right) + \frac{1}{2}t_2\left(\frac{t_2}{2m^2M} - 1\right) \\ &- \left(\frac{7M}{24} + \frac{1}{2}j\left(\frac{j}{M} - 1\right) + \frac{1}{2}(M-2j)\left(\frac{M-2j}{2M} - 1\right)\right).\end{aligned}$$

As regards the formulae for s at (2.17) and (2.18), it can be seen from (2.5) that converting the left side of (2.16) into the right side involves transformations of the form

$$(q^{-r}; q^k)_\infty (q^{k+r}; q^k)_\infty = (1 - q^{-r})(q^{k-r}; q^k)_\infty \frac{(q^r; q^k)_\infty}{1 - q^r} = -q^{-r}(q^r; q^k)_\infty (q^{k-r}; q^k)_\infty,$$

and thus that a factor of (-1) is introduced with each such operation, so that s is the number of negative exponents in the initial product on the left side of (2.16). We prove (2.17) only, as the argument for (2.18) is similar. On the left side of (2.16) we consider the + case (so each $\pm$ becomes + and each $\mp$ becomes $-$) and it is easily seen that three of the five exponents are positive for all values of r and the two that are possibly negative for some values of r are

$$-2jm + m^2M - \frac{1}{3}mM(m + 6r - 1), \qquad -jm + m^2M - \frac{1}{6}mM(m + 6r - 1).$$

Inequalities (2.17) simply reflect when none, one or both of these quantities are negative. □

In the remainder of this section, we briefly discuss the m-dissection for the infinite product $(q;q)_\infty$ that can be derived from Theorem 1.1.

In [8, Theorem 2.1.], the second author proved the m-dissection for $(q;q)_\infty$ in the next theorem. Proofs were previously given by Evans [4] and Ramanathan [9], but the method of proof in [8] was essentially that given by Berndt [1, Theorem 12.1, page 274]. Our formulation is slightly different.

Theorem 2.5. *Let $|q| < 1$ and $Q(z,q)$ be as at (1.3).*

(i) If m is a positive integer of the form $6t+1$, then

(2.19) $$(q;q)_\infty = (-1)^{(m-1)/6} q^{(m^2-1)/24} \left(q^{m^2}; q^{m^2}\right)_\infty + \sum_{u=0}^{\frac{m-1}{3}} (-1)^u q^{u(3u-1)/2} Q\left(-q^{m(m+6u-1)/6}; q^{m^2}\right) + \sum_{u=1}^{\frac{m-7}{6}} (-1)^u q^{u(3u+1)/2} Q\left(-q^{m(m-6u-1)/6}; q^{m^2}\right).$$

(ii) If m is a positive integer of the form $6t-1$, then

(2.20) $$(q;q)_\infty = (-1)^{(m+1)/6} q^{(m^2-1)/24} \left(q^{m^2}; q^{m^2}\right)_\infty$$

$$+\sum_{u=0}^{\frac{m-5}{6}}(-1)^u q^{u(3u-1)/2}Q\left(-q^{m(m-6u+1)/6};q^{m^2}\right)+\sum_{u=1}^{\frac{m-2}{3}}(-1)^u q^{u(3u+1)/2}Q\left(-q^{m(m+6u+1)/6};q^{m^2}\right).$$

In [8], the above theorem was proved independently of the proof given in the same paper of the m-dissection of $Q(z,q)$. This overlooked the elementary observation (noticed while conducting investigations for the present paper) that

$$Q(-1,q)=2(q;q)_\infty. \tag{2.21}$$

This leads to another proof of Theorem 2.5, after setting $z=-1$ in Theorem 1.1 (although each term in the expansions (2.19) and (2.20) appear twice, so that it is necessary to divide both sides by 2 to finally obtain Theorem 2.5).

We return now to the strategy mentioned at (2.21) of proving Theorem 2.5 by letting $z\to -1$ in Theorem 1.1. The advantage of this method of proof is that it also leads to expansions similar to those in Theorem 2.5 when m has either of the forms $m=6t+2$ or $m=6t+4$, for some non-negative integer t.

We note that this extends the result of Evans [4], Ramanathan [9] and Berndt [1], who proved the m-dissection of $(q;q)_\infty$ in terms of quintuple products, but limited to *odd* m relatively prime to 3.

Theorem 2.6. *Let* $|q|<1$ *and* $Q(z,q)$ *be as at* (1.3).
If m *is a positive integer of the form* $6t+2$*, then*

$$(q;q)_\infty =\sum_{u=0}^{\frac{m-2}{6}}(-1)^u q^{u(3u-1)/2}Q\left(q^{m(m-6u+1)/6};q^{m^2}\right)+\sum_{u=1}^{\frac{m-2}{3}}(-1)^u q^{u(3u+1)/2}Q\left(q^{m(m+6u+1)/6};q^{m^2}\right). \tag{2.22}$$

If m *is a positive integer of the form* $6t+4$*, then*

$$(q;q)_\infty =\sum_{u=0}^{\frac{m-1}{3}}(-1)^u q^{u(3u-1)/2}Q\left(q^{m(m+6u-1)/6};q^{m^2}\right)+\sum_{u=1}^{\frac{m-4}{6}}(-1)^u q^{u(3u+1)/2}Q\left(q^{m(m-6u-1)/6};q^{m^2}\right). \tag{2.23}$$

Proof. The method of proof is similar to the proof described for Theorem 2.5 (letting $z\to -1$ in Theorem 1.1), except that we take m to be even instead of odd. The details are omitted. □

We note that, unlike Theorem 2.5, Theorem 2.6 leads to $m/2$ dissections, not m-dissections. Most of the implications of Theorem 2.6 for $m=2^jm'$, with j a positive integer and $m'>1$ odd, are actually disguised versions of results that follow from Theorem 2.5 applied to m', via some elementary q-product manipulations. However, there is a family of results that follow from Theorem 2.6 that do not arise in this way. This is the case $m=2^j$ for some positive integer j. Section 4 addresses this case and proves Hirschhorn's conjecture for the 2^n-dissection of $(q;q)_\infty$.

3. Partition identities from the m-dissection of $(q;q)_\infty$ and other quintuple products

For use in this section we define two quite general partition functions. Firstly, for any set of positive integers S, define

(3.1) $D_S(n):=$ the number of partitions of n into an <u>even</u> number of distinct parts from S
minus the number of partitions of n into an <u>odd</u> number of distinct parts from S.

Secondly, for a positive integer $M \geq 5$ and a positive integer $a < M/2$ define, for any positive integer n,

(3.2) $P_{a,M}(n) :=$

$$\text{the number of partitions of } n \text{ into parts} \not\equiv \pm a, 0 \pmod{M}, \not\equiv M \pm 2a \pmod{2M}.$$

3.1. **Partition identities from the m-dissection of $(q;q)_\infty$.** Before proving Theorem 1.2, we need the following number-theoretic result.

Lemma 3.1. *When $m \equiv 1 \pmod 6$, there is no $0 \leq u \leq \frac{m-1}{3}$ such that*

$$\frac{u(3u-1)}{2} \equiv \frac{m^2-1}{24} \pmod{m}$$

if and only if m is square free. Likewise, there is no $0 \leq u \leq \frac{m-7}{6}$ such that

$$\frac{u(3u+1)}{2} \equiv \frac{m^2-1}{24} \pmod{m}$$

Proof. Notice that the congruences can be rewritten as

$$(6u \mp 1)^2 \equiv m^2 - 1 \pmod{24m},$$

so since $m \equiv 1 \pmod 6$, this congruence is solvable for u if and only if

$$(6u \mp 1)^2 \equiv 0 \pmod{m}$$

is solvable for u. If m is square free, one must have $6u \pm 1 \equiv 0 \pmod{m}$, but in the case regarding $6u-1$ this forces $6u-1=m$ under the assumption that $0 \leq u \leq \frac{m-1}{3}$, which contradicts the assumption of $m \equiv 1 \pmod 6$. The case regarding $6u+1$ is impossible given that $0 \leq u \leq \frac{m-7}{6}$. The sufficiency of m being square free follows.

On the other hand, assume that $m = \prod p^{e_p}$ is not square free, i.e., $e_p \geq 2$ for some $p|m$. Notice that any prime factor $p \geq 5$, and so

$$1, 5 \leq \frac{2m-1}{\prod_{p|m} p^{\lfloor \frac{e_p+1}{2} \rfloor}}, \ \frac{m-6}{\prod_{p|m} p^{\lfloor \frac{e_p+1}{2} \rfloor}},$$

since at least one of $\lfloor \frac{e_p+1}{2} \rfloor$ is strictly less than e_p, and $p \geq 5$. Thus, there are integers

$$k_- \leq \frac{2m-1}{\prod_{p|m} p^{\lfloor \frac{e_p+1}{2} \rfloor}} \text{ and } k_+ \leq \frac{m-6}{\prod_{p|m} p^{\lfloor \frac{e_p+1}{2} \rfloor}}$$

such that

$$\left(k_\mp \prod_{p|m} p^{\lfloor \frac{e_p+1}{2} \rfloor} \right) \pm 1 \equiv 0 \pmod 6,$$

since $\prod_{p|m} p^{\lfloor \frac{e_p+1}{2} \rfloor} \equiv 1, 5 \pmod 6$. Set $u_\mp = \frac{1}{6}\left(k_\mp \prod_{p|m} p^{\lfloor \frac{e_p+1}{2} \rfloor} \pm 1\right)$. It is clear by the fact that

$$k_\mp \leq \frac{2m-1}{\prod_{p|m} p^{\lfloor \frac{e_p+1}{2} \rfloor}}, \ \frac{m-6}{\prod_{p|m} p^{\lfloor \frac{e_p+1}{2} \rfloor}}$$

that $0 \leq u_- \leq \frac{m-1}{3}$, and $0 \leq u_+ \leq \frac{m-7}{6}$, and

$$(6u \mp 1)^2 = k_\mp^2 \prod_{p|m} p^{2\lfloor \frac{e_p+1}{2} \rfloor} \equiv 0 \pmod{m}.$$

Counterparts of the case $m \equiv -1 \pmod 6$ also hold and can be justified by the same argument. As such, we omit the details. ∎

As a consequence of Theorem 2.5 and Lemma 3.1, we obtain the general partition identities in Theorem 1.2 mentioned in the introduction, which have something of the flavour of (1.2). Recall that $b_m(n)$ is the number of m-regular partitions of n (partitions with no parts $\equiv 0 \pmod{m}$). In what follows, we give a proof of Theorem 1.2, and Since some of the statements in Theorem 1.2 are slightly complicated, we restate it here for the benefit of the reader.

Theorem 1.2. *Let $m \geq 5$ be an integer relatively prime to 6 and square-free, and let S be the set of positive integers containing no multiples of m. Define $D_S(n)$ to be number of partitions of n into an <u>even</u> number of distinct parts from S minus the number of partitions of n into an <u>odd</u> number of distinct parts from S.*

(i) Define $r = (m^2-1)/24$. If $m \equiv 1 \pmod{6}$ set $s = (m-1)/6$ and if $m \equiv -1 \pmod{6}$ set $s = (m+1)/6$. Then

$$D_S(mn+r) = (-1)^s b_m(n), \qquad \text{for all } n \geq 0. \tag{1.6}$$

(ii) If $m \equiv 1 \pmod{6}$, define

$$T := \left\{ \frac{u(3u-1)}{2} \middle| 0 \leq u \leq \frac{m-1}{3} \right\} \cup \left\{ \frac{u(3u+1)}{2} \middle| 1 \leq u \leq \frac{m-7}{6} \right\} \cup \left\{ \frac{m^2-1}{24} \right\} \pmod{m}, \tag{1.7}$$

and if $m \equiv 1 \pmod{6}$, define

$$T := \left\{ \frac{u(3u-1)}{2} \middle| 0 \leq u \leq \frac{m-5}{6} \right\} \cup \left\{ \frac{u(3u+1)}{2} \middle| 1 \leq u \leq \frac{m-2}{3} \right\} \cup \left\{ \frac{m^2-1}{24} \right\} \pmod{m}. \tag{1.8}$$

If $v \in \{0,1,2,\ldots,m-1\} \setminus T$, then $D_S(mn+v) = 0$, for all $n \geq 0$.

Proof. (i) Dividing both sides of (2.19) or (2.20) by $(q^m;q^m)$, the left side becomes

$$(q, q^2, \ldots, q^{m-1}; q^m)_\infty = \sum_{n=0}^{\infty} D_S(n) q^n. \tag{3.3}$$

On the right side of each equation one has by Lemma 3.1 that the equations $(m^2-1)/24 = u(3u-1)/2$ and $(m^2-1)/24 = u(3u+1)/2$ have no solutions in integers for u in the given summation ranges, so that with r and s as given,

$$(-1)^s q^r \frac{\left(q^{m^2}; q^{m^2}\right)_\infty}{(q^m;q^m)_\infty} = \sum_{n=0}^{\infty} D_S(mn+r) q^{mn+r}. \tag{3.4}$$

However, after cancelling a factor of q^r both sides and then making the replacement $q \to q^{1/m}$, equation (3.4) becomes

$$\sum_{n=0}^{\infty} D_S(mn+r) q^n = (-1)^s \frac{(q^m;q^m)_\infty}{(q;q)_\infty} = (-1)^s \sum_{n=0}^{\infty} b_m(n) q^n.$$

(ii) This is immediate, since it is clear from the right sides of (2.19) and (2.20) that for such an integer v, the coefficient of q^{mn+v} is equal to 0 for all positive integers n. □

Example 1. *Take $m = 7$, so $s = (7-1)/6 = 1$ and $r = (7^2-1)/24 = 2$. Let S be the set of positive integers which are not multiples of 7.*

(i) If we take $n = 13$, then $mn + r = 7(13) + 2 = 93$. There are 44530 partitions of 93 into an <u>even</u> number of distinct parts from S, and there are 44620 partitions of 93 into an <u>odd</u> number of distinct parts from S. Hence

$$D_S(93) = 44530 - 44620 = (-1)^1 90,$$

in agreement with (1.6), since $b_7(13) = 90$.

(ii) On the other hand, if we take $n = 13$ and $v = 3$, then $mn + v = 7(13) + 3 = 94$. There are 48239 partitions of 94 into an <u>even</u> number of distinct parts from S, and there are also 48239 partitions of 94 into an <u>odd</u> number of distinct parts from S. Hence

$$D_S(94) = 48239 - 48239 = 0,$$

in agreement with part (ii) of Theorem 1.2, since one easily computes from (1.7) *that* $T = \{0, 1, 2, 5\}$ *and* $v = 3 \notin T$.

Partition interpretations of the other terms in the m-dissections are not so straightforward, as they involve partitions in which some parts occur in two colours. We illustrate this with an example.

Example 2. *Let S denote the set of positive integers with no multiples of 5. By* (2.20),

$$\begin{aligned}
(q;q)_\infty &= -q\left(q^{25};q^{25}\right)_\infty \\
&\quad + \left(-q^5,-q^{20},q^{25};q^{25}\right)_\infty \left(q^{15},q^{35};q^{50}\right)_\infty - q^2\left(-q^{10},-q^{15},q^{25};q^{25}\right)_\infty \left(q^5,q^{45};q^{50}\right)_\infty \\
\Longrightarrow \frac{(q;q)_\infty}{(q^5;q^5)_\infty} &= -q\frac{\left(q^{25};q^{25}\right)_\infty}{(q^5;q^5)_\infty} \\
&\quad + \frac{\left(-q^5,-q^{20},q^{25};q^{25}\right)_\infty \left(q^{15},q^{35};q^{50}\right)_\infty}{(q^5;q^5)_\infty} - q^2\frac{\left(-q^{10},-q^{15},q^{25};q^{25}\right)_\infty \left(q^5,q^{45};q^{50}\right)_\infty}{(q^5;q^5)_\infty}
\end{aligned}$$

$$\Longrightarrow \frac{(q;q)_\infty}{(q^5;q^5)_\infty} = \sum_{n=0}^{\infty} D_S(n)q^n = -q\frac{\left(q^{25};q^{25}\right)_\infty}{(q^5;q^5)_\infty} + \frac{1}{\left(q^5,q^{20};q^{25}\right)^2_\infty} - \frac{q^2}{\left(q^{10},q^{15};q^{25}\right)^2_\infty},$$

where the last equality follows from elementary q-product manipulation. By proceeding as in earlier examples (comparing components with powers of q in the same arithmetic progressions modulo 5 on both sides of the last equation and then making the replacement $q \to q^{1/5}$, one gets that

$$\sum_{n=0}^{\infty} D_S(5n)q^n = \frac{1}{\left(q,q^4;q^5\right)^2_\infty}, \qquad \sum_{n=0}^{\infty} D_S(5n+1)q^n = -\frac{\left(q^5;q^5\right)_\infty}{(q;q)_\infty},$$

$$\sum_{n=0}^{\infty} D_S(5n+2)q^n = -\frac{1}{\left(q^2,q^3;q^5\right)^2_\infty}, \qquad \sum_{n=0}^{\infty} D_S(5n+3)q^n = \sum_{n=0}^{\infty} D_S(5n+4)q^n = 0.$$

Recall that $b_5(n)$ is the number of 5-regular partitions of n, and for $a \in \{1,2\}$, let $p_{a;5}(n)$ denote the number of partitions of n into parts $\equiv \pm a \pmod 5$, where parts come in two colours. Then the series-product identities above have the following partition interpretations:

- $D_S(5n) = p_{1;5}(n)$;
- $D_S(5n+1) = -b_5(n)$;
- $D_S(5n+2) = -p_{2;5}(n)$;
- *for any integer of the form $5n+3$ or $5n+4$, the number of partitions it has into an <u>even</u> number of distinct parts from S is the same as the number of partitions it has into an <u>odd</u> number of distinct parts from S.*

To give some specific numerical examples, $75 = 5(15)$ has 6140 partitions into an <u>even</u> number of distinct parts from S and 5944 partitions into an <u>odd</u> number of distinct parts from S, so that $D_S(75) = 6140 - 5944 = 196$, in agreement with $p_{1;5}(15) = 196$.

For a second example, $76 = 5(15) + 1$ has 6506 partitions into an <u>even</u> number of distinct parts from S and 6633 partitions into an <u>odd</u> number of distinct parts from S, so that $D_S(76) = 6506 - 6633 = -127 = -b_5(15)$.

Lastly, $78 = 5(15) + 3$ *has 7755 partitions into an* *<u>even</u>* *number of distinct parts from* S *and also 7755 partitions into an* *<u>odd</u>* *number of distinct parts from* S*, so that* $D_S(78) = 7755 - 7755 = 0$*, in agreement with the fourth entry in the bullet list above.*

Remark 2. One can similarly derive coloured partition identities from quintuple product expansions of $(q;q)_\infty$ for other values of m of the form $m = 6t \pm 1$ in Theorem 2.5, but they are not quite so straightforward, in that just some (not all) of the various arithmetic progressions have parts that come in two colours.

3.2. **Partition identities from the m-dissection of other quintuple products.** We next consider some partition implications of other quintuple products, as stated in Corollary 2.2.

Example 3. *Let* $P_{a,M}(n)$ *be as defined at* (3.2).

(i) Let S *be the set of positive integers* $\equiv \pm 1, \pm 3, \pm 4 \pmod{10}$*. Then*

$$\begin{aligned} D_S(5n) &= P_{4,25}(n) \\ D_S(5n+1) &= -P_{6,25}(n) \\ D_S(5n+2) &= P_{9,25}(n-1) \\ D_S(5n+3) &= -P_{1,25}(n) \\ D_S(5n+4) &= -P_{11,25}(n-2) \end{aligned} \tag{3.5}$$

(ii) Let S' *be the set of positive integers* $\equiv \pm 1, \pm 5, \pm 6 \pmod{14}$*. Then*

$$\begin{aligned} D_{S'}(7n) &= P_{8,49}(n) \\ D_{S'}(7n+1) &= -P_{6,49}(n) \\ D_{S'}(7n+2) &= -P_{20,49}(n-4) \\ D_{S'}(7n+3) &= P_{15,49}(n-1) \\ D_{S'}(7n+4) &= P_{1,49}(n-1) \\ D_{S'}(7n+5) &= -P_{13,49}(n) \\ D_{S'}(7n+6) &= P_{22,49}(n-5) \end{aligned} \tag{3.6}$$

(3.7)

Proof. (i) With the substitutions $p = m = 5$ and $j = 1$ in (2.14), one gets that

$$\begin{aligned} &\left(q, q^4, q^5; q^5\right)_\infty \left(q^3, q^7; q^{10}\right)_\infty = \left(q^{20}, q^{105}, q^{125}; q^{125}\right)_\infty \left(q^{85}, q^{165}; q^{250}\right)_\infty \\ &+ q^{122} \left(\frac{1}{q^{80}}, q^{205}, q^{125}; q^{125}\right)_\infty \left(\frac{1}{q^{35}}, q^{285}; q^{250}\right)_\infty + q^{69} \left(\frac{1}{q^{55}}, q^{180}, q^{125}; q^{125}\right)_\infty \left(q^{15}, q^{235}; q^{250}\right)_\infty \\ &+ q^{31} \left(\frac{1}{q^{30}}, q^{155}, q^{125}; q^{125}\right)_\infty \left(q^{65}, q^{185}; q^{250}\right)_\infty + q^{8} \left(\frac{1}{q^{5}}, q^{130}, q^{125}; q^{125}\right)_\infty \left(q^{115}, q^{135}; q^{250}\right)_\infty \\ &= \left(q^{20}, q^{105}, q^{125}; q^{125}\right)_\infty \left(q^{85}, q^{165}; q^{250}\right)_\infty + q^{7} \left(q^{45}, q^{80}, q^{125}; q^{125}\right)_\infty \left(q^{35}, q^{215}; q^{250}\right)_\infty \\ &- q^{14} \left(q^{55}, q^{70}, q^{125}; q^{125}\right)_\infty \left(q^{15}, q^{235}; q^{250}\right)_\infty - q \left(q^{30}, q^{95}, q^{125}; q^{125}\right)_\infty \left(q^{65}, q^{185}; q^{250}\right)_\infty \\ &- q^{3} \left(q^{5}, q^{120}, q^{125}; q^{125}\right)_\infty \left(q^{115}, q^{135}; q^{250}\right)_\infty, \end{aligned}$$

where the last equality follows from using (2.5). If we now divide both sides by $\left(q^5; q^5\right)_\infty$, the left side becomes

$$\left(q, q^4; q^5\right)_\infty \left(q^3, q^7; q^{10}\right)_\infty = \left(q, q^3, q^4, q^6, q^7, q^9; q^{10}\right)_\infty = \sum_{n=0}^{\infty} D_S(n) q^n,$$

where S is as defined at (i) above. If we consider, for example, powers of q with exponent $\equiv 4 \pmod 5$ on the resulting right side, we get

$$\begin{aligned}\sum_{n=0}^{\infty} D_S(5n+4)q^{5n+4} &= -q^{14}\frac{\left(q^{55},q^{70},q^{125};q^{125}\right)_\infty\left(q^{15},q^{235};q^{250}\right)_\infty}{(q^5;q^5)_\infty}\\ \implies \sum_{n=0}^{\infty} D_S(5n+4)q^{n} &= -q^{2}\frac{\left(q^{11},q^{14},q^{25};q^{25}\right)_\infty\left(q^{3},q^{47};q^{50}\right)_\infty}{(q;q)_\infty}\\ &= -q^2\sum_{n=0}^{\infty} P_{11,3;25}(n)q^n,\end{aligned}$$

and the last assertion at (3.5) now follows. The proofs of the other parts of (3.5) follow similarly, and are omitted.

(ii) Likewise, with the substitutions $p=m=7$ and $j=1$ in (2.15), one obtains

$$\begin{aligned}&\left(q,q^6,q^7;q^7\right)_\infty\left(q^5,q^9;q^{14}\right)_\infty = \left(q^{56},q^{287},q^{343};q^{343}\right)_\infty\left(q^{231},q^{455};q^{686}\right)_\infty\\ &+q^{375}\left(\frac{1}{q^7},q^{350},q^{343};q^{343}\right)_\infty\left(\frac{1}{q^{357}},q^{1043};q^{686}\right)_\infty+q^{260}\left(q^{42},q^{301},q^{343};q^{343}\right)_\infty\left(\frac{1}{q^{259}},q^{945};q^{686}\right)_\infty\\ &+q^{166}\left(q^{91},q^{252},q^{343};q^{343}\right)_\infty\left(\frac{1}{q^{161}},q^{847};q^{686}\right)_\infty+q^{93}\left(q^{140},q^{203},q^{343};q^{343}\right)_\infty\left(\frac{1}{q^{63}},q^{749};q^{686}\right)_\infty\\ &+q^{41}\left(q^{154},q^{189},q^{343};q^{343}\right)_\infty\left(q^{35},q^{651};q^{686}\right)_\infty+q^{10}\left(q^{105},q^{238},q^{343};q^{343}\right)_\infty\left(q^{133},q^{553};q^{686}\right)_\infty\\ &=\left(q^{56},q^{287},q^{343};q^{343}\right)_\infty\left(q^{231},q^{455};q^{686}\right)_\infty+q^{11}\left(q^{7},q^{336},q^{343};q^{343}\right)_\infty\left(q^{329},q^{357};q^{686}\right)_\infty\\ &-q\left(q^{42},q^{301},q^{343};q^{343}\right)_\infty\left(q^{259},q^{427};q^{686}\right)_\infty-q^5\left(q^{91},q^{252},q^{343};q^{343}\right)_\infty\left(q^{161},q^{525};q^{686}\right)_\infty\\ &-q^{30}\left(q^{140},q^{203},q^{343};q^{343}\right)_\infty\left(q^{63},q^{623};q^{686}\right)_\infty+q^{41}\left(q^{154},q^{189},q^{343};q^{343}\right)_\infty\left(q^{35},q^{651};q^{686}\right)_\infty\\ &+q^{10}\left(q^{105},q^{238},q^{343};q^{343}\right)_\infty\left(q^{133},q^{553};q^{686}\right)_\infty,\end{aligned}$$

where once again the last equality follows from using (2.5). The remainder of the proof mirrors the methods used in part (i), so the details are omitted. □

The second situation we consider is where $m=pp'$, where p is the prime occurring in Corollary 2.2, and $p'\neq 2,3$ is a prime different from p. We approach this situation with examples, by way of introducing the general situation.

Example 4. *Let $P_{a,M}(n)$ be as defined at (3.2).*

(i) Let S be the set of positive integers $\equiv \pm1,\pm3,\pm4,0,5 \pmod{10}$, but $\not\equiv 0 \pmod{35}$. Then

(3.8)
$$\begin{aligned}D_S(35n+6)&=-P_{56,175}(n-6)\\ D_S(35n+13)&=P_{49,175}(n-3)\\ D_S(35n+20)&=-P_{21,175}(n)\\ D_S(35n+27)&=P_{84,175}(n-25)\\ D_S(35n+34)&=P_{14,175}(n-1)\\ D_S(35n+r)&=0, r\equiv 2,4,5 \pmod 7\end{aligned}$$

(ii) Let S' be the set of positive integers $\equiv \pm1,\pm5,\pm6,0,7 \pmod{14}$, but $\not\equiv 0 \pmod{35}$. Then

(3.9)
$$\begin{aligned}D_{S'}(35n+3)&=-P_{15,245}(n-4)\\ D_{S'}(35n+8)&=P_{55,245}(n-1)\\ D_{S'}(35n+13)&=-P_{120,245}(n-38)\end{aligned}$$

$$
\begin{aligned}
D_{S'}(35n+18) &= -P_{50,245}(n)\\
D_{S'}(35n+23) &= P_{20,245}(n-2)\\
D_{S'}(35n+28) &= P_{90,245}(n-14)\\
D_{S'}(35n+33) &= -P_{85,245}(n-11)\\
D_{S'}(35n+r) &= 0, r\equiv 2,4 \pmod 5
\end{aligned}
$$

Remark 3. (1) Note that the first five statements at (3.8) collectively cover $D_S(7n+6)$, while the first seven statements at (3.9) collectively cover $D_{S'}(5n+3)$.

(2) In regard to the last items in (3.8) and (3.9), recall that $D_S(35n+r)=0$ (and likewise $D_{S'}(35n+r)=0$) means that the number of partitions of $35n+r$ composed of an even number of distinct parts from S is equal to the number of partitions of $35n+r$ composed of an odd number of distinct parts from S.

Proof. (i) Set $p=5$, $j=1$ and $m=35$ in (2.15) and then divide both sides by $(q^{35};q^{35})_\infty$, so that the left side becomes the generating function for the sequence $\{D_S(n)\}$. Upon computing the exponents $p(3r-1)r/2+3jr$ modulo 35, one concludes that

$$
\left\{\frac{5}{2}(3r-1)r+3r|r=0,1,\dots 34\right\} \pmod{35} = \{0,8,31,34,17,15,28,21,29,17,20,
$$
$$
3,1,14,7,15,3,6,24,22,0,28,1,24,27,10,8,21,14,22,10,13,31,29,7\}.
$$

From this list one notices two things. Firstly, the list contains none of the integers $2,4,5 \pmod 7$, which means no powers of q with such exponents are present on the right side (this is unchanged after dividing through by $(q^{35};q^{35})_\infty$), thus proving the final statement at (3.8). Secondly, the numbers which occur once in this list are 6, 13, 20, 27 and 34, so we consider only terms in the expansion resulting from (2.15) with powers of q in the arithmetic progressions $6,13,20,27,34 \pmod{35}$. The values of r which give rise to these five numbers are, respectively, $r=17,31,10,24$ and 3. We compute explicitly only the terms in the sum on the right side (2.15) corresponding to these r-values, and ignore the others. This yields

$$
\begin{aligned}
(q,q^4,q^5;q^5)_\infty (q^3,q^7;q^{10})_\infty &= \dots + q^{7223}\left(\frac{1}{q^{4410}},q^{10535},q^{6125};q^{6125}\right)_\infty\left(\frac{1}{q^{2695}},q^{14945};q^{12250}\right)_\infty\\
&+q^{4332}\left(\frac{1}{q^{3185}},q^{9310},q^{6125};q^{6125}\right)_\infty\left(\frac{1}{q^{245}},q^{12495};q^{12250}\right)_\infty\\
&+q^{2176}\left(\frac{1}{q^{1960}},q^{8085},q^{6125};q^{6125}\right)_\infty\left(q^{2205},q^{10045};q^{12250}\right)_\infty\\
&+q^{755}\left(\frac{1}{q^{735}},q^{6860},q^{6125};q^{6125}\right)_\infty\left(q^{4655},q^{7595};q^{12250}\right)_\infty\\
&+q^{69}\left(q^{490},q^{5635},q^{6125};q^{6125}\right)_\infty\left(q^{5145},q^{7105};q^{12250}\right)_\infty+\dots\\
&=\dots+q^{118}\left(q^{1715},q^{4410},q^{6125};q^{6125}\right)_\infty\left(q^{2695},q^{9555};q^{12250}\right)_\infty\\
&+q^{902}\left(q^{2940},q^{3185},q^{6125};q^{6125}\right)_\infty\left(q^{245},q^{12005};q^{12250}\right)_\infty\\
&-q^{216}\left(q^{1960},q^{4165},q^{6125};q^{6125}\right)_\infty\left(q^{2205},q^{10045};q^{12250}\right)_\infty\\
&-q^{20}\left(q^{735},q^{5390},q^{6125};q^{6125}\right)_\infty\left(q^{4655},q^{7595};q^{12250}\right)_\infty\\
&+q^{69}\left(q^{490},q^{5635},q^{6125};q^{6125}\right)_\infty\left(q^{5145},q^{7105};q^{12250}\right)_\infty+\dots,
\end{aligned}
$$

where, as previously, (2.5) has been used to get the second equality.

As mentioned above, after dividing both sides by $(q^{35};q^{35})_\infty$, the left side becomes the generating function for $D_S(n)$. Since $118 \equiv 13 \pmod{35}$, one gets that

$$\sum_{n=0}^{\infty} D_S(35n+13)q^{35n+13} = \frac{q^{118}\left(q^{1715},q^{4410},q^{6125};q^{6125}\right)_\infty \left(q^{2695},q^{9555};q^{12250}\right)_\infty}{(q^{35};q^{35})_\infty}$$
$$\implies \sum_{n=0}^{\infty} D_S(35n+13)q^{n} = q^3\frac{\left(q^{49},q^{126},q^{175};q^{175}\right)_\infty \left(q^{77},q^{273};q^{350}\right)_\infty}{(q;q)_\infty},$$

leading to the second statement at (3.8). The proofs of the remaining statements are similar, and so are omitted.

(ii) The starting point here is to set $p=7$, $j=1$ and $m=35$ in (2.15) and then again divide both sides by $(q^{35};q^{35})_\infty$, so that the left side becomes the generating function for the sequence $\{D_{S'}(n)\}$. Upon computing the exponents $p(3r-1)r/2+3jr$ modulo 35, one obtains

$$\left\{\frac{7}{2}(3r-1)r+3r|r=0,1,\dots 34\right\} \pmod{35} = \{0,10,6,23,26,15,25,21,3,6,30,5,1,18,$$
$$21,10,20,16,33,1,25,0,31,13,16,5,15,11,28,31,20,30,26,8,11\}.$$

This time one sees that the missing integers are those $\equiv 2,4 \pmod 5$, and those that occur exactly once are $3,8,13,18,23,28$ and 33. The rest of the proof parallels the proof of part (i), and so is omitted. ☐

4. Hirschhorn's conjecture on 2^n-dissection of $(q;q)_\infty$

In [5, page 332], Hirschhorn stated a conjecture for the 2^n-dissection of $(q;q)_\infty$. A proof by Gayan and Sarmah [10] of this conjecture by a different method has recently been published online.

As an application of Theorem 2.4, we provide a proof of Hirschhorn's conjecture, i.e., Theorem 1.3. Our proof is different from that given in [10]. For the reader's convenience, we restate the conjecture below before giving its proof.

Theorem 1.3. (*Hirschhorn's conjecture*) *Let $n \geq 1$ be an integer and let $m=2^n$. Then the m-dissection of $(q;q)_\infty$ is give by*

$$(1.9)\quad (q;q)_\infty = \sum_{k=1}^{m}(-1)^{k+\epsilon}q^{c_k}(q^{2(2k-1)m},q^{8m^2-2(2k-1)m};q^{8m^2})_\infty(q^{2m^2-(2k-1)m},q^{2m^2+(2k-1)m},q^{4m^2};q^{4m^2})_\infty,$$

where $\epsilon=0$ (respectively, 1) if n is odd (respectively, even), and for $k=1,2,3,\dots,2^n$,

$$(1.10)\qquad c_k = \begin{cases} P\left(\frac{2m-1}{3}-(k-1)\right), & \text{if } n \text{ is odd},\\ P\left(-\frac{2m-2}{3}+(k-1)\right), & \text{if } n \text{ is even},\end{cases}$$

where $P(t)=t(3t-1)/2$.

Proof. If n is odd, then $m=2^n=3t-1$, for some integer t, and likewise if n is even, then $m=2^n=3t+1$, for some integer t. Since $(q;q)_\infty=(q,q^3,q^4;q^4)_\infty(q^2,q^6;q^8)_\infty$, we will let $j=1$ and $p=4$ in Corollary 2.2 and use the two cases $m\equiv 1 \pmod 3$ and $m \equiv -1 \pmod 3$ to get initial m-dissections of $(q;q)_\infty$. These dissection will contain products with some negative exponents, so the next step is to use Theorem 2.4 to remove these negative exponents. Finally, a change in summation variable is used to show that the form of the m-dissection obtained after using Theorem 2.4 is in fact identical to that on the right side of (1.9). We will give the details for the case $m \equiv -1 \pmod 3$, and sketch the proof (which is similar) in the case $m \equiv 1 \pmod 3$.

As indicated above, we set $p = 4$ and $j = 1$ in (2.15) to get

$$(4.1) \quad (q;q)_\infty = \sum_{r=0}^{m-1} q^{r(6r+1)} \left(q^{m(2m-12r-1)/3}, q^{m(10m+12r+1)/3}, q^{4m^2}; q^{4m^2}\right)_\infty \times \left(q^{2m(8m-12r-1)/3}, q^{2m(4m+12r+1)/3}; q^{8m^2}\right)_\infty.$$

There are exactly two exponents in any particular term that may be negative for some values of r, namely $m(2m-12r-1)/3$ and $2m(8m-12r-1)/3$. Recall that the number of negative exponents (thus either zero, one or two) gives the values of the parameter s in (2.16), and also possibly changes the value of t_1 and/or t_2. To use Theorem 2.4 we divide the summation interval $0 \le r \le m-1$ into three sub-intervals in which none, exactly one or exactly two of these exponents are negative. It is easy to see from the formula for these two exponents that these 3 sub-intervals are, respectively,

$$0 \le r < \frac{2m-1}{12}, \qquad \frac{2m-1}{12} < r < \frac{8m-1}{12}, \qquad \frac{8m-1}{12} < r \le m-1.$$

In the first interval, both exponents are positive, so $s = 0$. From (4.1) and Theorem 2.4 one gets

$$t_1 = \frac{1}{3}m(2m-12r-1), \quad t_2 = \frac{2}{3}m(4m+12r+1),$$

so that $\mathcal{L} = r(6r+1)$. As expected, since all exponents are positive there is no change effected by Theorem 2.4. Upon comparing corresponding exponents in the q-products in (1.9) and (4.1), it can be seen that if the r-th term in (4.1) corresponds to the k-th term in (1.9), for some value of k, one must have

$$\begin{aligned} \frac{1}{3}m(2m-12r-1) &= 2m^2 - (2k-1)m, \\ \text{or } r &= \frac{1}{6}(3k-2m-2), \\ \text{or } k &= \frac{2(m+1)}{3} + 2r. \end{aligned}$$

Observe that $2(m+1)/3$ is even, and that the largest integer less than $(2m-1)/12$ is $(2m-4)/12$. Substituting this value in for r in the expression for k above, one gets $k = m$. Thus, the set of k-values corresponding to r-values in the interval $[0, (2m-4)/12]$ are the even integers in the interval $[2(m+1)/3, m]$. Further, a somewhat tedious check shows that after making the replacement for r indicated by the second equation above in the r-th term in (4.1), that it has the form of the k-term in (1.9) (since n is odd, $\epsilon = 0$, and since k is even, $(-1)^{k+\epsilon} = 1$). Thus the part of the sum at (4.1) corresponding to $0 \le r < (2m-1)/12$ is equal to the part of the sum at (1.9) over even k in the interval $[2(m+1)/3, m]$.

We next consider r-values in the second interval, $(2m-1)/12 < r < (8m-1)/12$. In this interval $m(-1+2m-12r)/3 < 0$, so $s = 1$ and

$$t_1 = \frac{1}{3}m(2m-12r-1) + 4m^2, \quad t_2 = \frac{2}{3}m(4m+12r+1),$$

so that $\mathcal{L} = \frac{1}{3}m(2m-12r-1) + 6r^2 + r$, and the r-th term in (4.1) becomes

$$(4.2) \quad -q^{m(2m-12r-1)/3+6r^2+r} \left(q^{m(14m-12r-1)/3}, q^{m(-2m+12r+1)/3}, q^{4m^2}; q^{4m^2}\right)_\infty \left(q^{2m(8m-12r-1)/3}, q^{2m(4m+12r+1)/3}; q^{8m^2}\right)_\infty$$

As above, it can be seen that if this equals a term in (1.9) for some value of k, one must have

$$\frac{1}{3}m(-2m+12r+1) = 2m^2-(2k-1)m,$$
$$\text{or } r = \frac{1}{6}(-3k+4m+1),$$
$$\text{or } k = \frac{1}{3}(1+4m)-2r.$$

It is an easy check that in the interval of r-values under consideration, $m(-2m+12r+1)/3 < m(14m-12r-1)/3$, which is why this smaller integer appears in the first equation. The smallest integer greater than $(2m-1)/12$ is $(2m+8)/12$ and the largest integer less than $(8m-1)/12$ is $(8m-4)/12$, and if the set of integers $(2m+8)/12 \leq r \leq (8m-4)/12$ are substituted into the formula for k above, one gets all the odd positive integers from 1 to $m-1$ inclusive. Upon substituting the value for r in the second equation above into (4.2), one gets the k-th term in the sum at (1.9) ($(-1)^{k+\epsilon}=-1$, since k is odd and $\epsilon=0$). Thus, this time the part of the sum at (4.1) corresponding to $(2m-1)/12 < r < (8m-1)/12$ is equal to the part of the sum at (1.9) over all odd k in the interval $[1, m-1]$.

Finally (in the case of $m \equiv -1 \pmod 3$), for the interval $(8m-1)/12 < r \leq m-1$, both $m(2m-12r-1)/3$ and $2m(8m-12r-1)/3$ are negative (hence $2m(4m+12r+1)/3 > 8m^2$), so $s=2$ and

$$t_1 = \frac{1}{3}m(2m-12r-1)+4m^2, \quad t_2 = \frac{2}{3}m(4m+12r+1)-8m^2,$$

so $\mathcal{L} = 6m^2-12mr-m+6r^2+r$, and the r-th term in (4.1) becomes

$$(4.3) \quad q^{6m^2-12mr-m+6r^2+r}$$
$$\times\left(q^{m(14m-12r-1)/3}, q^{m(-2m+12r+1)/3}, q^{4m^2}; q^{4m^2}\right)_\infty \left(q^{2m(20m-12r-1)/3}, q^{2m(-8m+12r+1)/3}; q^{8m^2}\right)_\infty.$$

In this r-interval $m(14m-12r-1)/3 < m(-2m+12r+1)/3$, so that if the term at (4.3) equals a term in (1.9) for some value of k, one must have

$$\frac{1}{3}m(14m-12r-1) = 2m^2-(2k-1)m,$$
$$\text{or } r = \frac{1}{6}(3k+4m-2),$$
$$\text{or } k = 2r-\frac{2}{3}(2m-1).$$

The smallest integer greater than $(8m-1)/12$ is $(8m+8)/12$ so that if the set of integers $(8m+8)/12 \leq r \leq m-1$ are substituted into the formula for k above, one gets the even positive integers in the interval $[2, 2(m-2)/3]$.

Note that this set of integers complements the even integers in $[2(m+1)/3, m]$ from the first r-interval $0 \leq r < (2m-1)/12$ to give all <u>even</u> integers k in the interval $[2, m]$. This in turn complements the set of <u>odd</u> integers k in the interval $[1, m-1]$ from the middle interval $(2m-1)/12 < r < (8m-1)/12$ to give <u>all</u> integers k in the interval $[1, m]$.

If one makes the substitution $r = (3k+4m-2)/6$ in (4.3) (noting that $(-1)^{k+\epsilon}=1$, since k is even and $\epsilon=0$) one gets the k-th term in (1.9) (a somewhat tedious check possibly most easily accomplished using a computer algebra system) so that taking into consideration the remarks in the previous paragraph, one finally has that the right side of (4.1) equals the right side of (1.9), thus completing the proof in the case $m \equiv -1 \pmod 3$.

The case $m \equiv 1 \pmod 3$ is similar, so we sketch the outline of the proof and omit the details. If the substitutions $p = 4$ and $j = 1$ are made in (2.14), one gets

$$(4.4) \quad (q;q)_\infty = \sum_{r=0}^{m-1} q^{r(6r+1)} \left(q^{m(2m+12r+1)/3}, q^{m(10m-12r-1)/3}, q^{4m^2}; q^{4m^2}\right)_\infty \times \left(q^{2m(8m+12r+1)/3}, q^{2m(4m-12r-1)/3}; q^{8m^2}\right)_\infty.$$

This time the exponents that can be negative for some values of r are $m(10m-12r-1)/3$ and $2m(4m-12r-1)/3$. The intervals where none, one or both of these are negative are

$$0 \le r < \frac{4m-1}{12}, \qquad \frac{4m-1}{12} < r < \frac{10m-1}{12}, \qquad \frac{10m-1}{12} < r \le m-1.$$

For these three intervals, respectively, one takes (t_1, t_2) to be the ordered pair

$$\left(\frac{1}{3}m(2m+12r+1),\ \frac{2}{3}m(8m+12r+1)\right),$$
$$\left(\frac{1}{3}m(2m+12r+1),\ \frac{2}{3}m(8m+12r+1) - 8m^2\right),$$
$$\left(\frac{1}{3}m(2m+12r+1) - 4m^2,\ \frac{2}{3}m(8m+12r+1) - 8m^2\right).$$

For these three intervals, respectively, one solves the equations

$$\frac{1}{3}m(2m+12r+1) = 2m^2 - (2k-1)m,$$
$$\frac{1}{3}m(10m-12r-1) = 2m^2 - (2k-1)m,$$
$$\frac{1}{3}m(-10m+12r+1) = 2m^2 - (2k-1)m.$$

The sets of k-values, respectively, corresponding to each of these r-intervals, respectively, are

$$k \in \left[1, \frac{2m+1}{3}\right],\ k \text{ odd},$$
$$k \in [2, m],\ k \text{ even},$$
$$k \in \left[\frac{2m+7}{3}, m-1\right],\ k \text{ odd}.$$

Note that the three collections of k-values above once again include exactly all the integers k in the interval $[1, m]$. Upon following through the steps as in the $m \equiv -1 \pmod 3$ case, one similarly finds that the right side of (4.4) equals the right side of (1.9), thus completing the proof in the case $m \equiv 1 \pmod 3$. $\square$

As an implication of Theorem 1.3, we next give an explicit determination of the sign of the coefficients in the series expansion of $(q;q)_\infty/(q^m;q^m)_\infty$ for $m = 2^n$ and show that the pattern of signs is periodic modulo m.

Corollary 4.1. *Let n be a positive integer and set $m = 2^n$. Define the sequence $\{d_n\}$ by*

$$(4.5) \qquad \frac{(q;q)_\infty}{(q^m;q^m)_\infty} =: \sum_{n=0}^{\infty} d_j q^j.$$

Then the signs of the d_j are periodic with period m, i.e. $d_j d_{j+m} \ge 0$ for all $j \ge 0$. Moreover, the sign of d_j may be determined explicitly as subsequently described.

Define

$$S = \{6v^2 + 3v | v = 0, 1, \ldots, m/2 - 1\} \pmod{m}.$$

For odd n,

$$d_j \begin{cases} \leq 0, & \textit{if } \left(j - \frac{2m^2+1}{3}\right) \pmod{m} \in S \\ \geq 0, & \textit{otherwise.} \end{cases} \tag{4.6}$$

For even n,

$$d_j \begin{cases} \geq 0, & \textit{if } \left(j - \frac{2m^2+1}{3}\right) \pmod{m} \in S \\ \leq 0, & \textit{otherwise.} \end{cases} \tag{4.7}$$

Proof. The m-dissection of $(q;q)_\infty/(q^m;q^m)_\infty$ follows from (1.9), and it is clear that no component is identically zero. Also each component has the form (see (1.9) for the meaning of the notation)

$$(-1)^{k+\epsilon} q^{c_k} \frac{(q^{2(2k-1)m}, q^{8m^2-2(2k-1)m}; q^{8m^2})_\infty (q^{2m^2-(2k-1)m}, q^{2m^2+(2k-1)m}, q^{4m^2}; q^{4m^2})_\infty}{(q^m;q^m)_\infty}$$
$$= (-1)^{k+\epsilon} q^{c_k} \frac{1}{\prod_{u \in T}(q^{um}; q^{8m^2})_\infty},$$

where

$$T = \{1, 2, \ldots 8m\} \backslash \{2(2k-1), 2m-(2k-1), 2m+(2k-1), 6m-(2k-1), 6m+(2k-1), 8m-2(2k-1)\}.$$

It is clear that all coefficients in the series expansion of the second infinite product have positive sign, thus that the coefficients in each component of the m-dissection all have the same sign, which is completely determined by $(-1)^{k+\epsilon}$ (so $\operatorname{sign}(d_{c_k}) = \operatorname{sign}(d_{mt+c_k}) = (-1)^{k+\epsilon}$ for all $t \geq 0$). This shows the claimed m-periodicity of the pattern of signs. What remains is to show how to determine whether $(-1)^{k+\epsilon}$ is positive or negative from the value of c_k. We first consider when n is odd (so $\epsilon = 0$ and $(-1)^{k+\epsilon} = (-1)^k$), so that from (1.10)

$$\begin{aligned}
c_k = P\left(\frac{2m-1}{3} - (k-1)\right) &= \frac{3k^2}{2} - 2km - \frac{3k}{2} + \frac{2m^2}{3} + m + \frac{1}{3} \\
\Longrightarrow c_k - \frac{2m^2+1}{3} &\equiv \frac{3k(k-1)}{2} \pmod{m} \\
\Longrightarrow c_k - \frac{2m^2+1}{3} &\equiv 6u^2 + 3u \pmod{m}, \text{ if } k = 2u+1, \\
\Longrightarrow \left(c_k - \frac{2m^2+1}{3}\right) &\pmod{m} \in S \text{ if } k \text{ is odd, or } (-1)^k = -1 \\
\Longrightarrow \left(c_k - \frac{2m^2+1}{3}\right) &\pmod{m} \in S \text{ if } \operatorname{sign}(d_{c_k}) = -1.
\end{aligned}$$

If $k = 2v$, some v, $0 \leq v \leq m/2 - 1$, then $3k(k-1)/2 = 6v^2 - 3v$ and it is an easy check that $(c_k - (2m^2+1)/3) \pmod{m} = 6v^2 - 3v \pmod{m} \notin S$, proving the claim for n odd.

The case for n even is similar, except this time $\epsilon = 1$ and $(-1)^{k+\epsilon} = (-1)^{k+1}$. Slightly curiously, $P((2m-1)/3-(k-1)) = P(-(2m-2)/3+(k-1))$ symbolically, and the argument follows through almost exactly the same, except the conclusion is

$$\left(c_k - \frac{2m^2+1}{3}\right) \pmod{m} \in S \Longleftrightarrow \operatorname{sign}(d_{c_k}) = 1,$$

since we are working with $(-1)^{k+1}$ instead of $(-1)^k$. □

We shall see more on sign patterns of the coefficients of infinite products in Section 5.

The expansion at (1.9) also implies various partition theoretic results, which we illustrate with an example.

Example 5. *Set $m = 8$ (or $n = 3$) in (1.9) to get*

$$\begin{aligned}(q;q)_\infty = &\\ &\left(q^{40},q^{216},q^{256};q^{256}\right)_\infty\left(q^{176},q^{336};q^{512}\right)_\infty - q\left(q^{56},q^{200},q^{256};q^{256}\right)_\infty\left(q^{144},q^{368};q^{512}\right)_\infty\\ &- q^2\left(q^{24},q^{232},q^{256};q^{256}\right)_\infty\left(q^{208},q^{304};q^{512}\right)_\infty - q^{35}\left(q^{120},q^{136},q^{256};q^{256}\right)_\infty\left(q^{16},q^{496};q^{512}\right)_\infty\\ &- q^{12}\left(q^{88},q^{168},q^{256};q^{256}\right)_\infty\left(q^{80},q^{432};q^{512}\right)_\infty + q^5\left(q^{72},q^{184},q^{256};q^{256}\right)_\infty\left(q^{112},q^{400};q^{512}\right)_\infty\\ &+ q^{22}\left(q^{104},q^{152},q^{256};q^{256}\right)_\infty\left(q^{48},q^{464};q^{512}\right)_\infty + q^7\left(q^{8},q^{248},q^{256};q^{256}\right)_\infty\left(q^{240},q^{272};q^{512}\right)_\infty.\end{aligned}$$

Let S be the set of positive integers containing no multiples of 8, so that dividing both sides of the above equation by $(q^8;q^8)_\infty$, one gets on the left side

$$\frac{(q;q)_\infty}{(q^8;q^8)_\infty} = \sum_{n=0}^{\infty} D_S(n)q^n.$$

After splitting the series into eight sub-series $\sum_{n=0}^{\infty} D_S(8n+r)q^{8n+r}$, $r = 0, 1, \ldots 7$, equating each with the corresponding component on the right side, cancelling q^r each side, and then making the replacement $q \to q^{1/8}$ on each side, one gets

$$\begin{aligned}\sum_{n=0}^{\infty} D_S(8n)q^n &= \frac{\left(q^5,q^{27},q^{32};q^{32}\right)_\infty\left(q^{22},q^{42};q^{64}\right)_\infty}{(q;q)_\infty},\\ \sum_{n=0}^{\infty} D_S(8n+1)q^n &= -\frac{\left(q^7,q^{25},q^{32};q^{32}\right)_\infty\left(q^{18},q^{46};q^{64}\right)_\infty}{(q;q)_\infty},\\ \sum_{n=0}^{\infty} D_S(8n+2)q^n &= -\frac{\left(q^3,q^{29},q^{32};q^{32}\right)_\infty\left(q^{26},q^{38};q^{64}\right)_\infty}{(q;q)_\infty},\\ \sum_{n=0}^{\infty} D_S(8n+3)q^n &= -q^4\frac{\left(q^{15},q^{17},q^{32};q^{32}\right)_\infty\left(q^{2},q^{62};q^{64}\right)_\infty}{(q;q)_\infty},\\ \sum_{n=0}^{\infty} D_S(8n+4)q^n &= -q\frac{\left(q^{11},q^{21},q^{32};q^{32}\right)_\infty\left(q^{10},q^{54};q^{64}\right)_\infty}{(q;q)_\infty},\\ \sum_{n=0}^{\infty} D_S(8n+5)q^n &= \frac{\left(q^9,q^{23},q^{32};q^{32}\right)_\infty\left(q^{14},q^{50};q^{64}\right)_\infty}{(q;q)_\infty},\\ \sum_{n=0}^{\infty} D_S(8n+6)q^n &= q^2\frac{\left(q^{13},q^{19},q^{32};q^{32}\right)_\infty\left(q^{6},q^{58};q^{64}\right)_\infty}{(q;q)_\infty},\\ \sum_{n=0}^{\infty} D_S(8n+7)q^n &= \frac{\left(q,q^{31},q^{32};q^{32}\right)_\infty\left(q^{30},q^{34};q^{64}\right)_\infty}{(q;q)_\infty}.\end{aligned}$$

Each of these equations gives rise to a family of partition identities. For an odd integer a, $1 \le a \le 15$, $P_{a,32}(n)$ is defined by (3.2) to equal the number of partitions of n into parts $\not\equiv \pm a, 0 \pmod{32}$, $\not\equiv 32-2a, 32+2a \pmod{64}$. If we interpret the q-products on the right side of each equation above as the generating function for a type of restricted partition function, these identities imply

$D_S(8n) = P_{5,32}(n)$ $\qquad$ $D_S(8n+1) = -P_{7,32}(n)$ $\qquad$ $D_S(8n+2) = -P_{3,32}(n)$

$D_S(8n+3) = P_{15,32}(n-4)$ $\quad D_S(8n+4) = -P_{11,32}(n-1)$ $\quad D_S(8n+5) = P_{9,32}(n)$
$D_S(8n+6) = P_{13,32}(n-2)$ $\quad D_S(8n+7) = P_{1,32}(n)$

To give a numerical example, $87 = 8(10)+7$ *has 32424 partitions into an* *even* *number of distinct parts from* S *and 32412 partitions into an* *odd* *number of distinct parts from* S*. Thus* $D_S(87) = 32424 - 32412 = 12$*, in agreement with* $P_{1,32}(10) = 12$*. This can be verified by listing the partitions counted by* $P_{1,32}(10)$*. In the case of* $n = 10$*, these are simply partitions with no part equal to 1:*

$10 \quad 8+2 \quad 7+3 \quad 6+4 \quad 6+2+2 \quad 5+5 \quad 5+3+2$
$4+4+2 \quad 4+3+3 \quad 4+2+2+2 \quad 3+3+2+2 \quad 2+2+2+2+2.$

5. Periodicity of sign changes in the series expansions of some other infinite products

In the introduction, it is noted that the m-dissection identity can be used to study the pattern of signs of the coefficients of infinite products, and Theorem 1.4 is displayed as a noteworthy example. Corollary 4.1 in Section 4 provides another notable consequence.

In this section we shall prove Theorem 1.4 and give some similar results for some other infinite products. Before that, we restate Theorem 1.4 for the convenience of the reader.

Theorem 1.4. *Let* $p > 3$ *be a prime. For* $k \geq 1$*, write*

$$\frac{(q^{2^{k-1}}; q^{2^{k-1}})_\infty}{(q^p; q^p)_\infty^2} = \sum_{n=0}^{\infty} a_n q^n.$$

Then

(1) if $p \equiv 1 \pmod 3$*, one has that*

$$a_n \begin{cases} > 0 & \text{if } n \equiv 3 \cdot 2^k r^2 + 2^{k-1} r \pmod p \text{ with } 0 \leq r < \frac{4(2p+1)-6}{24} \text{ or } \frac{4(5p+1)-6}{24} < r \leq p-1, \\ & \qquad \text{and } n \geq \mathcal{L}(r,k), \\ = 0 & \text{if } n \not\equiv 3 \cdot 2^k r^2 + 2^{k-1} r \pmod p, \\ < 0 & \text{if } n \equiv 3 \cdot 2^k r^2 + 2^{k-1} r \pmod p \text{ with } \frac{4(2p+1)-6}{24} < r < \frac{4(5p+1)-6}{24}, \\ & \qquad \text{and } n \geq \mathcal{L}(r,k), \end{cases}$$

(2) if $p \equiv -1 \pmod 3$*, one has that*

$$a_n \begin{cases} > 0 & \text{if } n \equiv 3 \cdot 2^k r^2 + 2^{k-1} r \pmod p \text{ with } 0 \leq r < \frac{4(p+1)-6}{24} \text{ or } \frac{4(4p+1)-6}{24} < r \leq p-1, \\ & \qquad \text{and } n \geq \mathcal{L}(r,k), \\ = 0 & \text{if } n \not\equiv 3 \cdot 2^k r^2 + 2^{k-1} r \pmod p, \\ < 0 & \text{if } n \equiv 3 \cdot 2^k r^2 + 2^{k-1} r \pmod p \text{ with } \frac{4(p+1)-6}{24} < r < \frac{4(4p+1)-6}{24}, \\ & \qquad \text{and } n \geq \mathcal{L}(r,k), \end{cases}$$

where $\mathcal{L}(r,k)$ *is defined by*

$$t_1 = t_1(r) = 2^{k+1} p(p + (6r-1))/6 + p \pmod{2^{k+1} p^2},$$
$$t_2 = t_2(r) = 2^{k+1} p^2 + 2^k p + 2^{k+1}(p + (6r-1))p/3 \pmod{2^{k+2} p^2},$$
$$\mathcal{L}(r,k) = \frac{7 \cdot 2^{k-1} p^2}{6} + \frac{1}{2} t_1 \left(\frac{t_1}{2^{k+1} p^2} - 1 \right) + \frac{1}{2} t_2 \left(\frac{t_2}{2^{k+2} p^2} - 1 \right) - \frac{2^k}{48}.$$

Remark 4. One shall see from the proof of Theorem 1.4 that in either case, the first and last sub-cases of r are mutually exclusive.

Proof. As the proofs of both cases are similar, we only present the proof of the case of $p \equiv 1 \pmod 3$. In Theorem 2.4, set $j = 2^{k-1}$, $M = 2^{k+1}$ and $m = p \equiv 1 \pmod 3$, p a prime, to get

$$\begin{aligned}
&\left(q^{2^{k-1}}, q^{2^{k+1}-2^{k-1}}, q^{2^{k+1}}; q^{2^{k+1}}\right)_\infty \left(q^{2^{k+1}-2\cdot 2^{k-1}}, q^{2^{k+1}+2\cdot 2^{k-1}}; q^{2^{k+2}}\right)_\infty \\
&= (q^{2^{k-1}}; q^{2^{k-1}})_\infty \\
&= \sum_{r=0}^{p-1} (-1)^{s(r)} q^{\mathcal{L}(r)} (q^{t_1(r)}, q^{2^{k+1}p^2-t_1(r)}, q^{2^{k+1}p^2}; q^{2^{k+1}p^2})_\infty (q^{t_2(r)}, q^{2^{k+2}p^2-t_2(r)}; q^{2^{k+2}p^2})_\infty,
\end{aligned}$$

with

$$\begin{aligned}
t_1 = t_1(r) &= 2^{k+1}p(p+(6r-1))/6 + p \pmod{2^{k+1}p^2}, \\
t_2 = t_2(r) &= 2^{k+1}p^2 + 2^k p + 2^{k+1}(p+(6r-1))p/3 \pmod{2^{k+2}p^2}, \\
\mathcal{L}(r) &= \frac{7\cdot 2^{k-1}p^2}{6} + \frac{1}{2}t_1\left(\frac{t_1}{2^{k+1}p^2} - 1\right) + \frac{1}{2}t_2\left(\frac{t_2}{2^{k+2}p^2} - 1\right) - \frac{2^k}{48},
\end{aligned}$$

and

$$s(r) = \begin{cases} 0, & \text{if } 0 \le r < \frac{4(2p+1)-6}{24}, \\ 1, & \text{if } \frac{4(2p+1)-6}{24} < r < \frac{4(5p+1)-6}{24}, \\ 2, & \text{if } r > \frac{4(5p+1)-6}{24}, \end{cases}$$

so that

$$\frac{(q;q)_\infty}{(q^p;q^p)_\infty} = \sum_{r=0}^{p-1} (-1)^{s(r)} q^{\mathcal{L}(r)} \frac{(q^{t_1(r)}, q^{2^{k+1}p^2-t_1(r)}, q^{2^{k+1}p^2}; q^{2^{k+1}p^2})_\infty (q^{t_2(r)}, q^{2^{k+2}p^2-t_2(r)}; q^{2^{k+2}p^2})_\infty}{(q^p;q^p)_\infty}.$$

Notice that

$$\frac{(q^{t_1(r)}, q^{2^{k+1}p^2-t_1(r)}, q^{2^{k+1}p^2}; q^{2^{k+1}p^2})_\infty (q^{t_2(r)}, q^{2^{k+2}p^2-t_2(r)}; q^{2^{k+2}p^2})_\infty}{(q^p;q^p)_\infty} = \prod_{\substack{1 \le \ell \le 2^{k+2}p \\ \ell \notin S(r)}} \frac{1}{(q^{p\ell}; q^{2^{k+2}p^2})_\infty}$$

has coefficients all nonnegative, where

$$S(r) = \left\{2^{k+1}p,\ 2^{k+2}p,\ t_1/p,\ (2^{k+1}p^2 \pm t_1)/p,\ (2^{k+2}p^2 - t_1)/p,\ t_2/p,\ (2^{k+2}p^2 - t_2)/p\right\}.$$

Next, it is not hard to see that

$$\mathcal{L}(r) \equiv 3\cdot 2^k r^2 + 2^{k-1} r \pmod p,$$

so that the terms of the form q^n with $n \equiv x \pmod p$ are involved only if $x \equiv 3\cdot 2^k r^2 + 2^{k-1} r \pmod p$. In other words, $a_n = 0$ if $n \not\equiv 3\cdot 2^k r^2 + 2^{k-1} r \pmod p$. Finally, it remains to show that for $\frac{4(2p+1)-6}{24} < r < \frac{4(5p+1)-6}{24}$, and $0 \le r' \le p-1$, $\mathcal{L}(r') \equiv \mathcal{L}(r) \pmod p$ if and only if $\frac{4(2p+1)-6}{24} < r' < \frac{4(5p+1)-6}{24}$. By examining the residue of $\mathcal{L}(r)$ modulo p, one can see that $\mathcal{L}(r') \equiv \mathcal{L}(r) \pmod p$ if and only if

$$3\cdot 2^k r^2 + 2^{k-1} r \equiv 3\cdot 2^k r'^2 + 2^{k-1} r' \pmod p$$

if and only if

$$r = r' \quad \text{or} \quad r' \equiv -r - \frac{1}{6} \pmod p.$$

Since $p \equiv 1 \pmod 6$ (equivalently, $p \equiv 1 \pmod 3$), suppose that $p = 6k+1$, so that $-\frac{1}{6} \equiv k \pmod p$, and $k = \frac{p-1}{6}$. Then by the assumption $\frac{4(2p+1)-6}{24} < r < \frac{4(5p+1)-6}{24}$, one can deduce that

$$\frac{4(2p+1)-6}{24} < r' = p - r + \frac{p-1}{6} < \frac{4(5p+1)-6}{24},$$

as desired. Hence, whenever $n \equiv 6r^2 + r \pmod p$ with $\frac{4(2p+1)-6}{24} < r < \frac{4(5p+1)-6}{24}$, $(-1)^{s(r)} = -1$, and thus $a_n \leq 0$, as well as whenever $n \equiv 6r^2 + r \pmod p$ with $0 \leq r < \frac{4(2p+1)-6}{24}$ or $\frac{4(5p+1)-6}{24} < r \leq p-1$, $(-1)^{s(r)} = 1$, and thus $a_n \geq 0$. Finally, since

$$\begin{aligned}
&\frac{(q^{t_1(r)}, q^{2^{k+1}p^2 - t_1(r)}, q^{2^{k+1}p^2}; q^{2^{k+1}p^2})_\infty (q^{t_2(r)}, q^{2^{k+2}p^2 - t_2(r)}; q^{2^{k+2}p^2})_\infty}{(q^p; q^p)_\infty^2} \\
&= \prod_{\substack{1 \leq \ell \leq 2^{k+2}p \\ \ell \notin S(r)}} \frac{1}{(q^{p\ell}; q^{2^{k+2}p^2})_\infty} \prod_{n=2}^{\infty} \frac{1}{(1-q^{pn})} \left(\frac{1}{1-q^p} \right) \\
&= \left(1 + \sum_{n=1}^{\infty} b_n q^n \right) (1 + q^p + q^{2p} + q^{3p} + \cdots),
\end{aligned}$$

with $b_n \geq 0$, we conclude that the coefficient of the term q^{pk} in the expansion is at least 1, so the conclusions hold for $n = \mathcal{L}(r) + pk$ with $k \geq 0$. □

Remark 5. The condition $n \geq \mathcal{L}(r,k)$ is necessary. In fact, there are primes p and integers n such that $n \equiv 6r^2 + r \pmod p$ but $a_n = 0$. For example, for $p = 19$ and $n = 3 \equiv 6 \cdot 5^2 + 5 \pmod{19}$, one can find that $a_3 = 0$ in the expansion of $\frac{(q;q)_\infty}{(q^{19};q^{19})_\infty^2}$.

Besides Corollary 1.1, another interesting result that can be inferred from Theorem 1.4 is the density of the zero coefficients in the expansion of $\frac{(q^{2^{k-1}};q^{2^{k-1}})_\infty}{(q^p;q^p)_\infty^2}$.

Corollary 5.1. *For any positive integer k and any prime $p > 3$, write*

$$\frac{(q^{2^{k-1}}; q^{2^{k-1}})_\infty}{(q^p; q^p)_\infty^2} = \sum_{n=0}^{\infty} a_n q^n.$$

Then

$$\lim_{X \to \infty} \frac{|\{n \leq X : a_n = 0\}|}{X} = \frac{p-1}{2p},$$

and thus, the infinite product is never lacunary.

Proof. By Theorem 1.4, one can see that

$$\begin{aligned}
\{n \leq X : a_n = 0\} &= \{n \leq X : n \not\equiv 3 \cdot 2^k r^2 + 2^{k-1} r \pmod p\} \\
&= \left\{ n \leq X : \left(\frac{2^{2k-2} + 3 \cdot 2^{k+2} n}{p} \right) = -1 \right\}.
\end{aligned}$$

It is straightforward to show that for $p > 3$, there are exactly $\frac{p-1}{2}$ elements n of $\mathbb{Z}/p\mathbb{Z}$ such that $\left(\frac{2^{2k-2} + 3 \cdot 2^{k+2} n}{p} \right) = -1$, and the conclusion follows. □

As a byproduct of the proof of Theorem 1.4, one can also say something about the sign periodicity of the coefficients of $(q^{2^{k-1}}; q^{2^{k-1}})_\infty / (q^p; q^p)_\infty$.

Theorem 5.1. *Write*

$$\frac{(q^{2^{k-1}};q^{2^{k-1}})_\infty}{(q^p;q^p)_\infty}=\sum_{n=0}^{\infty}a_nq^n.$$

Then

(1) if $p\equiv 1 \pmod 3$*, one has that*

$$a_n\begin{cases}\geq 0 & \textit{if } n\equiv 3\cdot 2^kr^2+2^{k-1}r \pmod p \textit{ with } 0\leq r<\frac{4(2p+1)-6}{24} \textit{ or } \frac{4(5p+1)-6}{24}<r\leq p-1,\\ =0 & \textit{if } n\not\equiv 3\cdot 2^kr^2+2^{k-1}r \pmod p,\\ \leq 0 & \textit{if } n\equiv 3\cdot 2^kr^2+2^{k-1}r \pmod p \textit{ with } \frac{4(2p+1)-6}{24}<r<\frac{4(5p+1)-6}{24},\end{cases}$$

(2) if $p\equiv -1 \pmod 3$*, one has that*

$$a_n\begin{cases}\geq 0 & \textit{if } n\equiv 3\cdot 2^kr^2+2^{k-1}r \pmod p \textit{ with } 0\leq r<\frac{4(p+1)-6}{24} \textit{ or } \frac{4(4p+1)-6}{24}<r\leq p-1,\\ =0 & \textit{if } n\not\equiv 3\cdot 2^kr^2+2^{k-1}r \pmod p,\\ \leq 0 & \textit{if } n\equiv 3\cdot 2^kr^2+2^{k-1}r \pmod p \textit{ with } \frac{4(p+1)-6}{24}<r<\frac{4(4p+1)-6}{24},\end{cases}$$

and thus, in either case one always has that $a_na_{n+p}\geq 0$*.*

The next result is an extension of some recent work of Bringmann et al. [2, Theorem 1.3], which concerns the infinite products $\frac{(q;q)_\infty(q^2;q^2)_\infty^{2k}}{(q^4;q^4)_\infty^{2k+1}}$ for $k\geq 1$.

Theorem 5.2. *For a positive integer* k*, write*

$$\frac{(q;q)_\infty(q^2;q^2)_\infty^{2k}}{(q^4;q^4)_\infty^{2k+1}}=\sum_{n=0}^{\infty}c_nq^n.$$

Then

$$c_n\begin{cases}\geq 0 & \textit{if } n\equiv 0,3 \pmod 4,\\ \leq 0 & \textit{if } n\equiv 1,2 \pmod 4.\end{cases}$$

In particular, when $k=1$*, both inequalities are strict, and the resulting conclusion recovers a recent result of Bringmann et al.* [2, Theorem 1.3]*: for*

$$\frac{(q;q)_\infty(q^2;q^2)_\infty^{2}}{(q^4;q^4)_\infty^{3}}=\sum_{n=0}^{\infty}c_nq^n,$$

one has that

$$c_n\begin{cases}> 0 & \textit{if } n\equiv 0,3 \pmod 4,\\ < 0 & \textit{if } n\equiv 1,2 \pmod 4.\end{cases}$$

Proof. Take $m=2$ in Theorem 1.3 to get

$$(q;q)_\infty=A_0(q)-qA_1(q),$$

where

$$A_0(q)=(q^{12},q^{20};q^{32})_\infty(q^2,q^{14},q^{16};q^{16})_\infty,$$
$$A_1(q)=(q^{4},q^{28};q^{32})_\infty(q^6,q^{10},q^{16};q^{16})_\infty,$$

and also take $m=2^2$ to get

$$(q;q)_\infty=B_0(q)-qB_1(q)-q^2B_2(q)+q^7B_3(q),$$

where

$$B_0(q)=(q^{40},q^{88};q^{128})_\infty(q^{12},q^{52},q^{64};q^{64})_\infty,$$

$$
\begin{aligned}
B_1(q) &= (q^{56}, q^{72}; q^{128})_\infty (q^4, q^{60}, q^{64}; q^{64})_\infty, \\
B_2(q) &= (q^{24}, q^{104}; q^{128})_\infty (q^{20}, q^{44}, q^{64}; q^{64})_\infty, \\
B_3(q) &= (q^{8}, q^{120}; q^{128})_\infty (q^{28}, q^{36}, q^{64}; q^{64})_\infty.
\end{aligned}
$$

Then one can deduce that

$$
\begin{aligned}
(q^2;q^2)_\infty^{2k} &= \sum_{m=0}^{2k} \binom{2k}{m} A_0(q^2)^{2k-m} \left(-q^2 A_1(q^2)\right)^m \\
&= \sum_{m=0}^{k} \binom{2k}{2m} q^{4m} A_0(q^2)^{2k-2m} A_1(q^2)^{2m} - q^2 \sum_{m=0}^{k-1} \binom{2k}{2m+1} q^{4m} A_0(q^2)^{2k-(2m+1)} A_1(q^2)^{2m+1},
\end{aligned}
$$

and subsequently find that

$$
\begin{aligned}
&(q;q)_\infty (q^2;q^2)_\infty^{2k} \\
&= \big(B_0(q) - qB_1(q) - q^2 B_2(q) + q^7 B_3(q)\big) \\
&\quad \times \left(\sum_{m=0}^{k} \binom{2k}{2m} q^{4m} A_0(q^2)^{2k-2m} A_1(q^2)^{2m} - q^2 \sum_{m=0}^{k-1} \binom{2k}{2m+1} q^{4m} A_0(q^2)^{2k-(2m+1)} A_1(q^2)^{2m+1} \right) \\
&= \left(\sum_{m=0}^{k} \binom{2k}{2m} q^{4m} A_0(q^2)^{2k-2m} A_1(q^2)^{2m} B_0(q) \right. \\
&\qquad\qquad \left. + q^4 \sum_{m=0}^{k-1} \binom{2k}{2m+1} q^{4m} A_0(q^2)^{2k-(2m+1)} A_1(q^2)^{2m+1} \right) \\
&\quad - q \left(\sum_{m=0}^{k} \binom{2k}{2m} q^{4m} A_0(q^2)^{2k-2m} A_1(q^2)^{2m} B_1(q) \right. \\
&\qquad\qquad \left. + q^8 \sum_{m=0}^{k-1} \binom{2k}{2m+1} q^{4m} A_0(q^2)^{2k-(2m+1)} A_1(q^2)^{2m+1} B_3(q) \right) \\
&\quad - q^2 \left(\sum_{m=0}^{k-1} \binom{2k}{2m+1} q^{4m} A_0(q^2)^{2k-(2m+1)} A_1(q^2)^{2m+1} B_0(q) \right. \\
&\qquad\qquad \left. + \sum_{m=0}^{k} \binom{2k}{2m} q^{4m} A_0(q^2)^{2k-2m} A_1(q^2)^{2m} B_2(q) \right) \\
&\quad + q^3 \left(q^4 \sum_{m=0}^{k} \binom{2k}{2m} q^{4m} A_0(q^2)^{2k-2m} A_1(q^2)^{2m} B_3(q) \right. \\
&\qquad\qquad \left. + \sum_{m=0}^{k-1} \binom{2k}{2m+1} q^{4m} A_0(q^2)^{2k-(2m+1)} A_1(q^2)^{2m+1} B_2(q) \right).
\end{aligned}
$$

The conclusion follows from this dissection together with the simple fact that the coefficients of

$$
\frac{A_0(q^2)^{2k-m} A_1(q^2)^m B_i(q)}{(q^4;q^4)_\infty^{2k+1}}
$$

are all non-negative.

Finally, in the case of $k = 1$, one has that

$$\begin{aligned}(q;q)_\infty(q^2;q^2)_\infty^2 &= \left(B_0(q) - qB_1(q) - q^2B_2(q) + q^7B_3(q)\right)\left(A_0(q^2) - q^2A_1(q^2)\right)^2 \\ &= \left(2q^4A_0(q^2)A_1(q^2)B_2(q) + q^4A_1(q^2)^2B_0(q) + A_0(q^2)^2B_0(q)\right) \\ &\quad - q\left(A_0(q^2)^2B_1(q) + q^4A_1(q^2)^2B_1(q) + 2q^8A_0(q^2)A_1(q^2)B_3(q)\right) \\ &\quad - q^2\left(A_0(q^2)^2B_2(q) + 2A_0(q^2)A_1(q^2)B_0(q) + q^4A_1(q^2)^2B_2(q)\right) \\ &\quad + q^3\left(q^8A_1(q^2)^2B_3(q) + q^4A_0(q^2)^2B_3(q) + 2A_0(q^2)A_1(q^2)B_1(q)\right).\end{aligned}$$

Note that for $i = 0, 1, 2, 3$ and any $n \geq 0$, the coefficients of q^{4n} in

$$\frac{A_0(q^2)A_1(q^2)B_i(q)}{(q^4;q^4)_\infty^3}$$

are all at least 1. Together with this observation and the above dissection for $(q;q)_\infty(q^2;q^2)_\infty^2$, it follows that the conclusion holds for $n \geq 9$. Verifying the remaining finitely many cases completes the proof. □

6. Concluding Remarks

The initial impetus for the present work was the investigation of a certain type of partition identity. This in turn led us to the main result of this paper, which was the m-dissection of the general quintuple product $Q(z, q)$ for m relatively prime to 3 stated in Theorem 1.1.

We have shown that this m-dissection has a number of interesting implications. These include the extension of the result of Evans [4], Ramanathan [9] and Berndt [1] concerning the m-dissection of $(q;q)_\infty$ in terms of quintuple products. Our work extended the result from odd m relatively prime to 3 to include even m relatively prime to 3. Theorem 1.1 also allowed us to give a proof of Hirschhorn's conjecture concerning the 2^n-dissection of $(q;q)_\infty$. Upon specializing $Q(z, q)$ to $Q(q^j, q^M)$ for various integers j and M, we were led to a number of interesting partition identities. Similar specializations led to explicit statements concerning the pattern of sign changes of the coefficients in the series expansions of various eta quotients.

We hope that these discoveries may be useful to other investigators and lead to other relevant problems, for example, is there a combinatorial proof for the partition identity

$$D_S(mn + r) = (-1)^s b_m(n)$$

given in Theorem 1.2? We conclude by remarking that it might be intriguing to find combinatorial proofs of some of the partition identities in the present paper (see the remark following Theorem 1.2 for an outline of a possible general strategy for proving such identities). This can be a topic for future investigation.

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