A COUNTABLE SPACE WITH AN UNCOUNTABLE FUNDAMENTAL GROUP

JEREMY BRAZAS AND LUIS MATOS

ABSTRACT. Traditional examples of spaces that have uncountable fundamental group (such as the Hawaiian earring space) are path-connected compact metric spaces with uncountably many points. In this paper, we construct a T_0 compact, path-connected, locally path-connected topological space H with countably many points but with an uncountable fundamental group. The construction of H, which we call the "coarse Hawaiian earring" is based on the construction of the usual Hawaiian earring space $\mathbb{H} = \bigcup_{n\geq 1} C_n$ where each circle C_n is replace with a copy of the four-point "finite circle."

1. Introduction

Since fundamental groups are defined in terms of maps from the unit interval $[0, 1]$, students are often surprised to learn that spaces with finitely many points can be path-connected and have non-trivial fundamental groups. In fact, it has been known since the 1960s that the homotopy theory of finite spaces is quite rich [10, 14]. The algebraic topology has finite topological spaces has gained significant interest since Peter May's sequence of REU Notes [7, 8, 9]. See [2, 3, 5] and the references therein for more recent theory and applications.

While it is reasonable to expect that all finite connected spaces have finitely generated fundamental groups, it is rather remarkable that for every finitely generated group G one can construct a finite space X so that $\pi_1(X, x_0) \cong G$. In fact, every finite simplicial complex is weakly homotopy equivalent to a finite space [10]. In the same spirit, we consider the consider fundamental groups of spaces with coarse topologies.

It is well-known that there are connected, locally path-connected compact metric space whose fundamental groups are uncountable [4]. Since finite spaces can only have finitely generated fundamental groups, we must extend our view to spaces with countably many points. In this paper, we prove the following theorem.

Theorem 1. *There exists a connected, locally path-connected, compact, T_0 topological space H with countably many points such that $\pi_1(H, w_0)$ is uncountable.*

To construct such a space, we must consider examples which are not even locally finite, that is, spaces which have a point such that every neighborhood of that point contains infinitey many other points. Additionally, since our example must be path-connected, the following lemma demands that such a space cannot have the T_1 separation axiom.

Lemma 2. *Every countable T_1 space is totally path-disconnected.*

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Proof. If X is countable and T_1 and $\alpha : [0,1] \to X$ is a non-constant path, then $\{\alpha^{-1}(x) | x \in X\}$ is a non-trivial, countable partition of $[0,1]$ into closed sets. However, it is a classical result in general topology that such a partition of $[0,1]$ is impossible [12]. □

Ultimately, we construct the space H by modeling the construction of the traditional Hawaiian earring space $\mathbb{H}$, which is the prototypical space that fails to admit a universal covering space. The fundamental group of the Hawaiian earring is an uncountable group which plays a key role in Eda's homotopy classification of one-dimensional Peano continua [6]. Due to the similarities between $\mathbb{H}$ and H, we call H the *coarse Hawaiian earring*.

2. Fundamental Groups

Let X be a topological space with basepoint $x_0 \in X$. A *path* in X is a continuous function $\alpha : [0,1] \to X$. We say X is *path-connected* if every pair of points $x, y \in X$ can be connected by a path $p : [0,1] \to X$ with $p(0) = x$ and $p(1) = y$. All spaces in this paper will be path-connected.

We say a path p is a *loop* based at x_0 if $\alpha(0) = \alpha(1)$. Let $\Omega(X, x_0)$ be the set of continuous functions $p : [0,1] \to X$ such that $p(0) = p(1) = x_0$. Let $\alpha^- : [0,1] \to X$ be the reverse path of α defined as $\alpha^-(t) = \alpha(1-t)$. If α and β are paths in X satisfying $\alpha(1) = \beta(0)$, let $\alpha \cdot \beta$ be the concatenation defined piecewise as

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(2t), & 0 \le t \le \frac{1}{2} \\ \beta(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

More generally, if $\alpha_1, ..., \alpha_n$ is a sequence of paths such that $\alpha_i(1) = \alpha_{i+1}(0)$ for $i = 1, ..., n-1$, let $\prod_{i=1}^{n} \alpha_i$ be the path defined as α_i on the interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$.

Two loops α and β based at x_0 are *homotopic* if there is a map $H : [0,1] \times [0,1] \to X$ such that $H(s,0) = \alpha(s)$, $H(s,1) = \beta(s)$ and $H(0,t) = H(1,t) = x_0$ for all $s, t \in [0,1]$. We write $\alpha \simeq \beta$ if α and β are homotopic. Homotopy $\simeq$ is an equivalence relation on the set of loops $\Omega(X, x_0)$. The equivalence class $[\alpha]$ of a loop α is called the homotopy class of α. The set of homotopy classes $\pi_1(X, x_0) = \Omega(X, x_0)/\simeq$ is called the *fundamental group* of X at x_0. It is a group when it has multiplication $[\alpha] * [\beta] = [\alpha \cdot \beta]$ and $[\alpha]^{-1} = [\alpha^-]$ is the inverse of $[\alpha]$ [11]. A space X is *simply connected* if X is path-connected and $\pi_1(X, x_0)$ is isomorphic to the trivial group. Finally, a map $f : X \to Y$ such that $f(x_0) = y_0$ induces a well-defined homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $f_*([\alpha]) = [f \circ \alpha]$.

Fundamental groups are often studied using maps called covering maps. We refer to [11] and [13] for the basic theory of algebraic topology.

Definition 3. Let $p : \widetilde{X} \to X$ be a map. An open set $U \subseteq X$ is *evenly covered* by p if $p^{-1}(U) \subseteq \widetilde{X}$ is the disjoint union $\coprod_{\lambda \in \Lambda} V_\lambda$ where V_λ is open in $\widetilde{X}$ and $p|_{V_\lambda} : V_\lambda \to U$ is a homeomorphism for every $\lambda \in \Lambda$. A covering map is a map $p : \widetilde{X} \to X$ such that every point $x \in X$ has an open neighborhood which is evenly covered by p. We call p a universal covering map if $\widetilde{X}$ is simply connected.

An important property of covering maps is that for every path $\alpha : [0,1] \to X$ such that $\alpha(0) = x_0$ and point $y \in p^{-1}(x_0)$ there is a unique path $\widetilde{\alpha}_y : [0,1] \to \widetilde{X}$ (called a *lift* of α) such that $p \circ \widetilde{\alpha}_y = p$ and $\widetilde{\alpha}_y(0) = y$.

Lemma 4. [11] *A covering map $p : \widetilde{X} \to X$ such that $p(y) = x_0$ induces an injective homomorphism $p_* : \pi_1(\widetilde{X}, y) \to \pi_1(X, x_0)$. If $\alpha : [0, 1] \to X$ is a loop based at x_0, then $[\alpha] \in p_*(\pi_1(\widetilde{X}, y))$ if and only if $\widetilde{\alpha}_y(1) = y$.*

A covering map $p : \widetilde{X} \to X$ induces a lifting correspondence map $\phi : \pi_1(X, x_0) \to p^{-1}(x_0)$ from the fundamental group of X to the fiber over x_0. This is defined by the formula $\phi([\alpha]) = \widetilde{\alpha}_y(1)$.

Lemma 5. [11] *If $p : \widetilde{X} \to X$ is a covering map, then the lifting correspondence $\phi : \pi_1(X, x_0) \to p^{-1}(x_0)$ is surjective. If p is a universal covering map, then p is bijective.*

Example 6. Let $S^1 = \{(x, y)|x^2 + y^2 = 1\}$ be the unit circle and $b_0 = (1, 0)$. The exponential map $\epsilon : \mathbb{R} \to S^1$, $\epsilon(t) = (\cos(2\pi t), \sin(2\pi t))$ defined on the real line is a covering map such that $\epsilon^{-1}(0) = \mathbb{Z}$ is the set of integers. The lifting correspondence for this covering map $\phi : \pi_1(S^1, b_0) \to \epsilon^{-1}(b_0) = \mathbb{Z}$ is an isomorphism when $\mathbb{Z}$ is the additive group of integers.

3. Some basic finite spaces

A *finite space* is a topological space $X = \{x_1, x_2, ..., x_n\}$ with finitely many points.

Example 7. The *coarse interval* is the three-point space $I = \{0, 1/2, 1\}$ with topology generated by the basic sets the sets $\{0\}$, $\{1\}$, and I. In other words, the topology of I is $T_I = \{I, \{0\}, \{1\}, \{0, 1\}, \emptyset\}$.

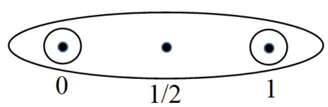

Figure 1. The coarse interval I. A basic open set is illustrated here as a bounded region whose interior contains the points of the set.

The coarse interval is clearly T_0. It is also path-connected since we can define a continuous surjection $p : [0, 1] \to I$ by

$$p(t) = \begin{cases} 0, & t \in [0, 1/2) \\ 1/2 & t = 1/2 \\ 1, & t \in (1/2, 1] \end{cases}$$

and the continuous image of a path-connected space is path-connected. A space X is contractible if the identity map $id : X \to X$ is homotopic to a constant map $X \to X$. Every contractible space is simply connected.

Lemma 8. *The coarse interval I is contractible.*

Proof. To show I is contractible we define a continuous map $G : I \times [0, 1] \to I$ such that $G(x, 0) = x$ for $x \in I$ and $G(x, 1) = 1/2$. The set $C = (\{0, 1\} \times [1/2, 1]) \cup (\{1/2\} \times [0, 1])$ is closed in $I \times [0, 1]$. Define G by

$$G(s, t) = \begin{cases} 0, & (s, t) \in \{0\} \times [0, 1/2) \\ 1/2, & (s, t) \in C \\ 1, & (s, t) \in \{1\} \times [0, 1/2) \end{cases}$$

This function is well-defined and continuous since $\{0\}$ and $\{1\}$ are open in I. □

Corollary 9. *I is simply connected.*

For $n = 0, 1, 2, 3$, let $b_n = \left(\cos\left(\frac{n\pi}{2}\right), \cos\left(\frac{n\pi}{2}\right)\right) \in S^1$ be the points of the unit circle on the coordinate axes, i.e. $b_0 = (1, 0)$, $b_1 = (0, 1)$, $b_2 = (-1, 0)$, and $b_3 = (0, -1)$.

Example 10. The *coarse circle* is the four-point set $S = \{b_i | i = 0, 1, 2, 3\}$ with the topology generated by the basic sets $\{b_0, b_1, b_2\}$, $\{b_2, b_3, b_0\}$, $\{b_0\}$, and $\{b_2\}$. The entire topology of S may be written as $T_S = \{S, \{b_0, b_1, b_2\}, \{b_2, b_3, b_0\}, \{b_0, b_2\}, \{b_0\}, \{b_2\}, \emptyset\}$.

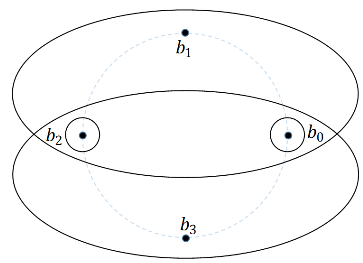

FIGURE 2. The coarse circle S and it's basic open sets.

Observe that the open sets $U_1 = \{b_0, b_1, b_2\}$ and $U_2 = \{b_2, b_3, b_0\}$ are homeomorphic to I when they are given the subspace topology. Since S is the union of two path-connected subsets with non-empty intersection, S is also path-connected.

4. The coarse line as a covering space

The main purpose of this section is to show that the fundamental group $\pi_1(S, b_0)$ of the coarse circle is isomorphic to the infinite cyclic group $\mathbb{Z}$, i.e. the additive group of integers.

Example 11. The *coarse line* is the set $L = \{\frac{n}{4} \in \mathbb{R} | n \in \mathbb{Z}\}$ with the topology generated by the basis $\mathscr{B}$ consisting of the sets $A_n = \{\frac{n}{2}\}$ and $B_n = \{\frac{n}{2}, \frac{2n+1}{4}, \frac{n+1}{2}\}$ for each $n \in \mathbb{Z}$. Even though L is not a finite space, it is a countable space with a T_0 but non-T_1 topology.

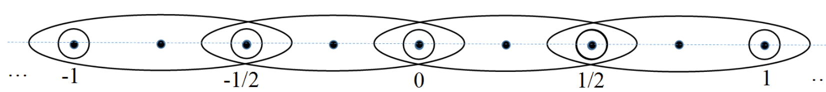

FIGURE 3. The basic open sets generating the topology of the coarse line L.

Lemma 12. *L is simply connected.*

Proof. The set $L_n = L \cap \left[-\frac{n}{2}, \frac{n}{2}\right]$ is open in L since it is the union of the basic sets $B_k = \{\frac{k}{2}, \frac{2k+1}{4}, \frac{k+1}{2}\}$, $k = -n, ..., n-1$ with the subspace topology of L. We prove using induction that L_n is simply connected. Since $B_k \cong I$ for each k, B_k is simply connected for each k. Observe that $L_1 = B_{-1} \cup B_0$ where $B_{-1} \cap B_0 = \{0\}$ is simply connected since it only has one point. By the van Kampen Theorem [11], L_1 is simply connected. Now suppose L_n is simply connected. Since L_n, B_n, and

$L_n \cap B_n = \{\frac{n}{2}\}$ are all simply connected, $L_n \cup B_n$ is simply connected by the van Kampen theorem. Similarly, since $L_n \cup B_n$, B_{-n-1}, and $(L_n \cup B_n) \cap B_{-k-1} = \{-\frac{n}{2}\}$ are all simply connected, $L_{n+1} = B_{-n-1} \cup L_n \cup B_n$ is simply connected by the van Kampen theorem. This completes the proof by induction.

Since L is the union of the path-connected sets L_n all of which contain 0, it follows that L is path-connected. Now suppose $\alpha : [0,1] \to L$ is a path such that $\alpha(0) = \alpha(1)$. Since $[0,1]$ is compact, the image $\alpha([0,1])$ is compact. But $\{L_n | n \geq 1\}$ is an open cover of L such that $L_n \subseteq L_{n+1}$. Since α must have image in a finite subcover of $\{L_n | n \geq 1\}$, we must have $\alpha([0,1]) \subseteq L_n$ for some n. But L_n is simply connected, showing that α is homotopic to the constant loop at 0. This proves $\pi_1(L, 0)$ is the trivial group, i.e. L is simply connected. □

Just like the usual covering map $\epsilon : \mathbb{R} \to S^1$ used to compute $\pi_1(S^1, b_0)$, we define a similar covering map in the coarse situation.

Example 13. Consider the function $p : L \to S$ from the coarse line to the coarse circle, which is the restriction of the covering map $\epsilon : \mathbb{R} \to S^1$. More directly, define $p\left(\frac{n}{4}\right) = b_{n \bmod 4}$. We check that each non-empty, open set in S can be written as a union of basic open sets in L. Since

- $p^{-1}(\{b_0\}) = \mathbb{Z} = \bigcup_{k \in \mathbb{Z}} A_{2k}$
- $p^{-1}(\{b_2\}) = \frac{1}{2} + \mathbb{Z} = \bigcup_{k \in \mathbb{Z}} A_{2k+1}$
- $p^{-1}(U_1) = \bigcup_{k \in \mathbb{Z}} B_{2k}$
- $p^{-1}(U_2) = \bigcup_{k \in \mathbb{Z}} B_{2k+1}$

we can conclude that p is continuous.

Lemma 14. *$p : L \to S$ is a covering map.*

Proof. We claim that the sets U_1, U_2 are evenly covered by p. Notice that $p^{-1}(U_1) = \bigcup_{k \in \mathbb{Z}} B_{2k}$ is a disjoint union where each B_{2k} is open. Recall that both B_{2k} and U_1 are homeomorphic to I; specifically $p|_{B_{2k}} : B_{2k} \to U_1$ is a homeomorphism. Thus U_1 is evenly covered. Similarly, $p^{-1}(U_2)$ is the disjoint union $\bigcup_{k \in \mathbb{Z}} B_{2k+1}$ where each B_{2k+1} is open and is mapped homeomorphically on to U_2 by p. □

Since $p : L \to S$ is a covering map and L is simply connected, p is a universal covering map. The proof of the following theorem is similar to the proof that the lifting correspondence for ϵ is a group isomorphism. We remark that even though L is not a topological group, for each $n \in \mathbb{Z}$, the shift map $\sigma_n : L \to L$, $\sigma(t) = t + n$ is a homeomorphism satisfying $p \circ \sigma_n = p$.

Theorem 15. *The lifting correspondence $\phi : \pi_1(S, b_0) \to p^{-1}(b_0) = \mathbb{Z}$ is a group isomorphism where $\mathbb{Z}$ has the usual additive group structure.*

Proof. Since $p : L \to S$ is a covering map and L is simply connected, ϕ is bijective by Lemma 5. Suppose $\alpha, \beta : [0,1] \to S$ are loops based at b_0. Respectively, let $\widetilde{\alpha}_0 : [0,1] \to L$ and $\widetilde{\beta}_0 : [0,1] \to L$ be the unique lifts of α and β starting at 0. Since $\widetilde{\alpha}_0(1) \in p^{-1}(b_0) = \mathbb{Z}$, we have $\phi([\alpha]) = \widetilde{\alpha}_0(1) = n$ for some integer n. Similarly, $\phi([\beta]) = \widetilde{\beta}_0(1) = m$ for some integer m.

Consider the concatenated path $\gamma = \widetilde{\alpha}_0 \cdot (\sigma_n \circ \widetilde{\beta_0}) : [0,1] \to L$ from 0 to $m+n$. Since $p \circ \sigma_n = p$, we have

$$\begin{aligned} p \circ \gamma &= p \circ (\widetilde{\alpha}_0 \cdot (\sigma_n \circ \widetilde{\beta_0})) \\ &= (p \circ \widetilde{\alpha}_0) \cdot (p \circ \sigma_n \circ \widetilde{\beta_0}) \\ &= (p \circ \widetilde{\alpha}_0) \cdot (p \circ \widetilde{\beta_0}) \\ &= \alpha \cdot \beta \end{aligned}$$

which means that γ is a lift of $\alpha \cdot \beta$ starting at 0. Since lifts are unique, this means $\gamma = \widetilde{\alpha \cdot \beta}_0$. It follows that $\phi([\alpha][\beta]) = \phi([\alpha \cdot \beta]) = \widetilde{\alpha \cdot \beta}_0(1) = \gamma(1) = m+n$. This proves ϕ is a group homomorphism. □

Both $\pi_1(S^1, b_0)$ and $\pi_1(S, b_0)$ are isomorphic to the infinite cyclic group $\mathbb{Z}$. In fact, we can define maps which induce the isomorphism between the two fundamental groups.

Let $f : \mathbb{R} \to L$ be the map defined so that $f\left(\left(\frac{n}{2} - \frac{1}{4}, \frac{n}{2} + \frac{1}{4}\right)\right) = \frac{n}{2}$ and $f\left(\frac{n}{2} + \frac{1}{4}\right) = \frac{n}{2} + \frac{1}{4}$ for each $n \in \mathbb{Z}$. Notice that $p \circ f$ is constant on each fiber $\epsilon^{-1}(x)$, $x \in S^1$. Therefore, there is an induced map $g : S^1 \to S$ such that $g \circ \epsilon = p \circ f$.

Proposition 16. *The induced homomorphism $g_* : \pi_1(S^1, b_0) \to \pi_1(S, b_0)$ is a group isomorphism.*

Proof. Recall that $\epsilon^{-1}(b_0) = \mathbb{Z}$ and $p^{-1}(b_0) = \mathbb{Z}$ and notice that the restriction to the fibers $f|_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ is the identity map. Let $i : [0,1] \to \mathbb{R}$ be the inclusion and note $f \circ i : [0,1] \to L$ is a path from 0 to 1. The group $\pi_1(S^1, b_0)$ is freely generated by the homotopy class of $\alpha = \epsilon \circ i$ and $\pi_1(S, b_0)$ is freely generated by the homotopy class of $p \circ f \circ i$. Since $g_*([\epsilon \circ i]) = [g \circ \epsilon \circ i] = [p \circ f \circ i]$, g_* maps one free generator to the other and it follows that g_* is an isomorphism. □

5. The coarse Hawaiian earring

Let $C_n = \left\{(x,y) \in \mathbb{R}^2 | (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\right\}$ be the circle of radius $\frac{1}{n}$ centered at $\left(\frac{1}{n}, 0\right)$. The *Hawaiian earring* is the countably infinite union $\mathbb{H} = \bigcup_{n \geq 1} C_n$ of these circles over the positive integers. We model this construction by replacing the usual circle with the coarse circle studied in the previous sections.

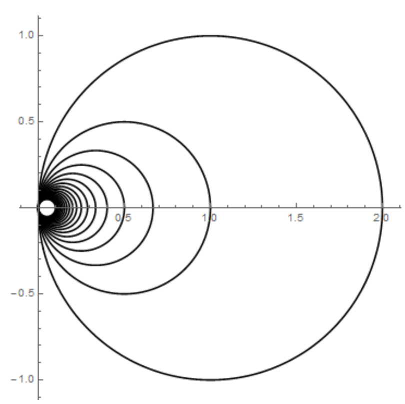

Figure 4. The Hawaiian earring $\mathbb{H}$

Let $w_0 = (0,0)$, and for integers $n \geq 1$ define $x_n = \left(\frac{1}{n}, -\frac{1}{n}\right)$, $y_n = \left(\frac{2}{n}, 0\right)$, and $z_n = \left(\frac{1}{n}, \frac{1}{n}\right)$. Let $D_n = \{w_0, x_n, y_n, z_n\}$ and $H = \bigcup_{n\geq 1} D_n$. Note that H is a countable subset of $\mathbb{H}$.

Definition 17. The *coarse Hawaiian earring* is the set H with the topology generated by the basis consisting of the following sets for each $n \geq 1$: $\{x_n\}$, $\{z_n\}$, $\{x_n, y_n, z_n\}$, and $V_n = \bigcup_{j\geq n} D_j \cup \{x_n | n \geq 1\} \cup \{z_n | n \geq 1\}$.

Notice that $V_n = H \backslash \{y_1, ..., y_{n-1}\}$ contains all but finitely many of the finite circles D_j. These sets form a neighborhood base at w_0 so that H is not an Alexandrov-discrete space in the sense of [1].

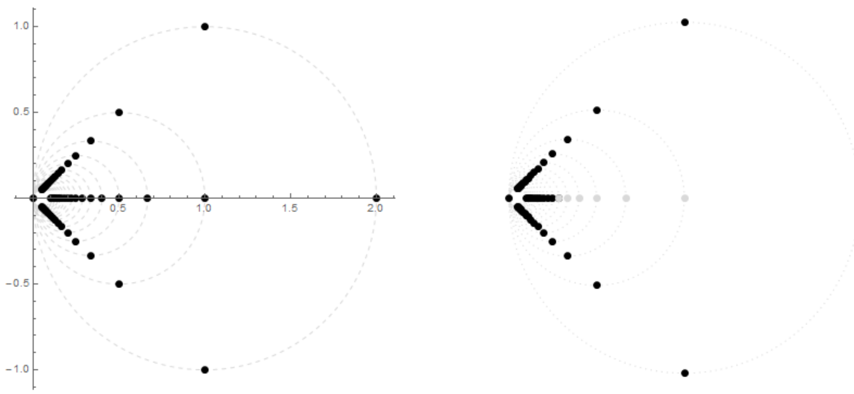

FIGURE 5. The underlying set of the coarse Hawaiian earring H (left) and the basic open set V_8 (right).

Proposition 18. *H is a path-connected, locally path-connected, compact, T_0 space which is not T_1.*

Proof. Notice that the set $D_n \subset H$ is homeomorphic to the coarse circle S when equipped with the subspace topology of H. Since D_n is path-connected and $w_0 \in D_n$ for each $n \geq 1$, it follows that H is path-connected.

To see that H is locally path-connected, we check that every basic open set is path-connected. Certainly, $\{x_n\}$ and $\{z_n\}$ are path-connected. Since $\{x_n, y_n, z_n\}$ is homeomorphic to I when it is given the subspace topology of H, this basic open set is path-connected. Additionally, the subspace $\{w_0, x_n, y_n\} \subseteq H$ is homeomorphic to I and is path-connected. Therefore, since V_n is the union $\bigcup_{j\geq n} D_n \cup \bigcup_{n\geq 1}\{w_0, x_n, y_n\}$ of sets all of which are path-connected and contain w_0, we can conclude that V_n is path-connected. This proves H is locally path-connected.

To see that H is compact let $\mathscr{U}$ be an open cover of H. Since the only basic open sets containing w_0 are the sets V_n there must be a $U_0 \in \mathscr{U}$ such that $w_0 \in V_n \subseteq U_0$ for some n. For $i = 1, ..., n-1$, find a set $U_i \in \mathscr{U}$ such that $y_i \in U_i$. Now $\{U_0, U_1, ..., U_{n-1}\}$ is a finite subcover of $\mathscr{U}$. This proves H is compact.

To see that H is T_0, we pick two points $a, b \in H$. If $a = w_0$ and $b = y_n$, then $a \in V_{n+1}$ but $b \notin V_{n+1}$. If $a = w_0$ and $b \in \{x_n, z_n\}$, then b lies in the open set $\{x_n, y_n, z_n\}$ but a does not. If $a \in \{x_n, z_n\}$ and $a \neq b$, then $\{a\}$ is open and does not contain b. This concludes all the possible cases of distinct pairs of points in H proving that H is T_0.

H is not T_1 since the every open neighborhood V_n of w_0 contains the infinite set $\bigcup_{n\geq 1}\{w_0, x_n, z_n\}$. □

Since $D_n \cong S$, by Theorem 15, we have $\pi_1(D_n, w_0) \cong \mathbb{Z}$ for each $n \geq 1$. Recall that if A is a subspace of X, then a retraction is a map $r : X \to A$ such that $r|_A : A \to A$ is the identity map.

Proposition 19. *For each $n \geq 1$, the function $r_n : H \to D_n$ which is the identity on D_n and collapses $\bigcup_{j\neq n} D_j$ to w_0 is a retraction.*

Proof. Since D_n is a subspace of H it suffices to show r_n is continuous. We have $r_n^{-1}(\{x_n\}) = \{x_n\}$, $r_n^{-1}(\{z_n\}) = \{z_n\}$, $r_n^{-1}(\{x_n, y_n, z_n\}) = \{x_n, y_n, z_n\}$, and $r_n^{-1}(\{w_0, x_n, y_n\}) = \{x_n\} \cup \{y_n\} \cup V_{n+1} \cup \bigcup_{j<n}\{x_j, y_j, z_j\}$. Since the pullback of each basic open set in D_n is the union of basic open sets in H, r_n is continuous. □

Corollary 20. *H is not semi-locally simply connected at w_0.*

Proof. Fix $n \geq 1$. We show that V_n contains a loop α which is not null-homotopic in H. Let $\alpha : [0, 1] \to D_n$ be any loop based at w_0 such that $[\alpha]$ is not the identity element of $\pi_1(D_n, w_0)$. Let $i : D_n \to H$ be the inclusion map so that $r_n \circ i = id_{D_n}$ is the identity map. Since π_1 is a functor, $(r_n)_* \circ i_* = (r_n \circ i)_* = id_{\pi_1(D_n,w_0)}$ is the identity homomorphism of $\pi_1(D_n, w_0)$. In particular, $i \circ \alpha$ is a loop in H with image in $D_n \subseteq V_n$ such that $(r_n)_*([i \circ \alpha]) = [\alpha]$ is not the identity of $\pi_1(D_n, w_0)$. Since homomorphisms preserve identity elements, $[i \circ \alpha]$ cannot be the identity element of $\pi_1(H, w_0)$. □

Definition 21. The *infinite product* of a sequence of groups $G_1, G_2, ...$ is denoted $\prod_{n\geq 1} G_n$ and consists of all infinite sequences $(g_1, g_2, ...)$ with $g_n \in G_n$ for each $n \geq 1$. Group multiplication and inversion are evaluated component-wise. If $G_n = \mathbb{Z}$ for each $n \geq 1$, then the group $\prod_{n\geq 1} \mathbb{Z}$ consisting of sequences $(n_1, n_2, ...)$ of integers is called the *Baer-Specker group*.

Infinite products of groups have the useful property that if G is a fixed group and $f_n : G \to G_n$ is a sequence of homomorphisms, then there is a well-defined homomorphism $f : G \to \prod_{n\geq 1} G_n$ given by $f(g) = (f_1(g), f_2(g), ...)$.

Lemma 22. *The infinite product $\prod_{n\geq 1} \pi_1(D_n, w_0)$ is uncountable.*

Proof. If each G_n is non-trivial, then G_n contains at least two elements. Therefore the product $\prod_{n\geq 1} G_n$ is uncountable since the Cantor set $\{0, 1\}^{\mathbb{N}} = \prod_{n\geq 1}\{0, 1\}$ can be injected as a subset. In particular, the Baer-Specker group is uncountable. Since $\pi_1(D_n, w_0) \cong \mathbb{Z}$ for each $n \geq 1$, the infinite product $\prod_{n\geq 1} \pi_1(D_n, w_0)$ is isomorphic to the Baer-Specker group and is therefore uncountable. □

Let $\lambda_n : [0, 1] \to D_n$ be the loop defined as

$$\lambda_n(t) = \begin{cases} w_0, & t \in \{0, 1\} \\ x_n, & t \in (0, 1/2) \\ y_n, & t = 1/2 \\ z_n, & t \in (1/2, 1) \end{cases}$$

This function is continuous and therefore a loop in D_n. In particular, our description of the universal covering of S in the previous section shows that the homotopy class $[\lambda_n]$ is a generator of the cyclic group $\pi_1(D_n, b_0)$.

Definition 23. Suppose for each $n \geq 1$, we have a continuous loop $\alpha_n : [0,1] \to H$ based at w_0 with image in D_n. The *infinite concatenation* of this sequence of loops is the loop $\alpha_\infty : [0,1] \to H$ defined as follows: for each $n \geq 1$, the restriction of α_∞ to $\left[\frac{n-1}{n}, \frac{n}{n+1}\right]$ is the path α_n and $\alpha_\infty(1) = w_0$.

Lemma 24. *α_∞ is a continuous loop such that $[r_n \circ \alpha_\infty] = [\alpha_n]$ for each $n \geq 1$.*

Proof. Since each loop α_n is continuous and each concatenation $\alpha_n \cdot \alpha_{n+1}$ is continuous, it is enough to show that α_∞ is continuous at 1. Consider a basic open neighborhood V_n of $\alpha_\infty(1) = w_0$. Since α_i has image in V_n for each $i \geq n$, we have $\alpha_\infty\left(\left[\frac{n-1}{n}, 1\right]\right) \subseteq V_n$. In particular, $1 \in \left(\frac{n-1}{n}, 1\right] \subseteq f^{-1}(V_n)$. This proves that α_∞ is continuous.

Since $r_n \circ \alpha_\infty$ is defined as α_n on $\left[\frac{n-1}{n}, \frac{n}{n+1}\right]$ and is constant at w_0 on $\left[0, \frac{n-1}{n}\right] \cup \left[\frac{n}{n+1}, 1\right]$ (and the suitable arrangement when $n = 1$. Therefore $r_n \circ \alpha_\infty$ is homotopic to α_n. □

Theorem 25. *$\pi_1(H, w_0)$ is uncountable.*

Proof. We have a sequence of homomorphisms $(r_n)_* : \pi_1(H, w_0) \to \pi_1(D_n, w_0)$. Together these induce a homomorphism $r : \pi_1(H, w_0) \to \prod_{n \geq 1} \pi_1(D_n, w_0)$. given by

$$r([\alpha]) = ((r_1)_*([\alpha]), (r_2)_*([\alpha]), ...) = ([r_1 \circ \alpha], [r_2 \circ \alpha], ...).$$

By Lemma 22, the infinite product $\prod_{n \geq 1} \pi_1(D_n, w_0)$ is uncountable. We claim that r is onto.

Suppose $(g_1, g_2, ...) \in \prod_{n \geq 1} \pi_1(D_n, w_0)$ where $g_n \in \pi_1(D_n, w_0)$. Since g_n is an element of the infinite cyclic group $\pi_1(D_n, w_0)$ generated by $[\lambda_n]$, we may write $g_n = [\lambda_n]^{m_n}$ for some integer $m_n \in \mathbb{Z}$.

For each $n \geq 1$, define a loop α_n as follows:

$$\alpha_n = \begin{cases} \prod_{i=1}^{m_n} \lambda_n, & \text{if } m_n > 0 \\ \text{constant at } w_0, & \text{if } m_n = 0 \\ \prod_{i=1}^{|m_n|} \lambda_n^-, & \text{if } m_n < 0 \end{cases}$$

Notice that α_n is defined so that $g_n = [\lambda_n]^{m_n} = [\alpha_n]$. Let $\alpha_\infty : [0,1] \to H$ be the loop based at w_0 which is the infinite concatenation as in Definition 23. By Lemma 24, we have $[r \circ \alpha_\infty] = [\alpha_n] = g_n$ for each $n \geq 1$. Therefore $r([\alpha_\infty]) = (g_1, g_2, ...)$. This proves that r is onto.

Since $\pi_1(H, b_0)$ surjects onto an uncountable group, it must also be uncountable. □

We conclude that there is a T_0 space with countably many points but which has an uncountable fundamental group.

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