Elements of higher homotopy groups undetectable by polyhedral approximation

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Abstract

When non-trivial local structures are present in a topological space X, a common approach to characterizing the isomorphism type of the n-th homotopy group $\pi_n(X,x_0)$ is to consider the image of $\pi_n(X,x_0)$ in the n-th Čech homotopy group $\check{\pi}_n(X,x_0)$ under the canonical homomorphism $\Psi_n : \pi_n(X,x_0) \to \check{\pi}_n(X,x_0)$. The subgroup $\ker(\Psi_n)$ is the obstruction to this tactic as it consists of precisely those elements of $\pi_n(X,x_0)$, which cannot be detected by polyhedral approximations to X. In this paper, we use higher dimensional analogues of Spanier groups to characterize $\ker(\Psi_n)$. In particular, we prove that if X is paracompact, Hausdorff, and UV^{n-1}, then $\ker(\Psi_n)$ is equal to the n-th Spanier group of X. We also use the perspective of higher Spanier groups to generalize a theorem of Kozlowski-Segal, which gives conditions ensuring that Ψ_n is an isomorphism.

1 Introduction

When non-trivial local structures are present in a topological space, a common approach to characterizing the isomorphism type of $\pi_n(X,x_0)$ is to consider the image of $\pi_n(X,x_0)$ in the n-th Čech (shape) homotopy group $\check{\pi}_n(X,x_0)$ under the canonical homomorphism $\Psi_n : \pi_n(X,x_0) \to \check{\pi}_n(X,x_0)$. The *n-th shape kernel* $\ker(\Psi_n)$ is the obstruction to this tactic as it consists of precisely those elements of $\pi_n(X,x_0)$, which cannot be detected by polyhedral approximations to X. This method has proved successful in many situations for both the fundamental group [5, 11, 15, 17] and higher homotopy groups [3, 12, 13, 14, 21]. In this paper, we study the map Ψ_n and give a characterization the n-th shape kernel in terms of higher-dimensional analogues of Spanier groups.

The subgroups of fundamental groups, which are now commonly referred to as "Spanier groups," first appeared in E.H. Spanier's unique approach to covering space theory [30]. If $\mathscr{U}$ is an open cover of a topological space X and $x_0 \in X$, then the *Spanier group with respect to* $\mathscr{U}$ is the subgroup $\pi_1^{Sp}(\mathscr{U},x_0)$ of $\pi_1(X,x_0)$ generated by path-conjugates $[\alpha][\gamma][\alpha]^{-1}$ where α is a path starting at x_0 and γ is a loop based at $\alpha(1)$ with image in some element of $\mathscr{U}$. These subgroups are particularly relevant to covering space theory since, when X is locally path-connected, a subgroup $H \leqslant \pi_1(X, x_0)$ corresponds to a covering map $p : (Y, y_0) \to (X, x_0)$ if and only if $\pi_1^{Sp}(\mathscr{U}, x_0) \leqslant H$ for some open cover $\mathscr{U}$ [30, 2.5.12]. The intersection $\pi_1^{Sp}(X, x_0) = \bigcap_{\mathscr{U}} \pi_1^{Sp}(\mathscr{U}, x_0)$ is called the *Spanier group of* (X, x_0) [16]. The inclusion $\pi_1^{Sp}(X, x_0) \subseteq \ker(\Psi_1)$ always holds [18, Prop. 4.8]. It is proved in [4, Theorem 6.1] that $\pi_1^{Sp}(X, x_0) = \ker(\Psi_1)$ whenever X is paracompact Hausdorff and locally path connected. The upshot of this equality is having a description of level-wise generators (for each open cover $\mathscr{U}$) whereas there may be no readily available generating set for the kernel of a homomorphism induced by a canonical map from X to the nerve $|N(\mathscr{U})|$. Indeed, 1-dimensional Spanier groups have proved useful in persistence theory [32]. Since much of applied topology is based on a geometric refinement of polyhedral approximation from shape theory, there seems potential for higher dimensional analogues to be useful as well.

Higher dimensional analogues of Spanier groups recently appeared in [1] and are defined in a similar way: $\pi_n^{Sp}(\mathscr{U}, x_0)$ is the subgroup of $\pi_n(X, x_0)$ consisting of homotopy classes of path-conjugates $\alpha * f$ where α is a path starting at x_0 and $f : S^n \to X$ is based at $\alpha(1)$ with image in some element of $\mathscr{U}$. Then $\pi_n^{Sp}(X, x_0)$ is the intersection of these subgroups. In this paper, we prove a higher-dimensional analogue of the 1-dimensional equality $\pi_1^{Sp}(X, x_0) = \ker(\Psi_1)$ from [4].

A space X is UV^n if for every neighborhood U of a point $x \in X$, there is a neighborhood V of x in U such that every map $f : S^k \to V$, $0 \leqslant k \leqslant n$ is null-homotopic in U, c.f. [29]. When a space is UV^n "small" maps on spheres of dimension $\leqslant n$ contract by null-homotopies of relatively the same size. Certainly, every locally n-connected space is UV^n. However, when $n \geqslant 1$, the converse is not true even for metrizable spaces. Our main result is the following.

Theorem 1.1. *Let $n \geqslant 1$ and $x_0 \in X$. If X is paracompact, Hausdorff, and UV^{n-1}, then $\pi_n^{Sp}(X, x_0) = \ker(\Psi_n)$.*

This result confirms that higher Spanier groups, like their 1-dimensional counterparts, often identify precisely those elements of $\pi_n(X, x_0)$ which can be detected by polyhedral approximations to X. A first countable path-connected space is UV^0 if and only if it is locally path connected. Hence, in dimension $n = 1$, Theorem 1.1 only expands [4, Theorem 6.1] to some non-first countable spaces.

Regarding the proof of Theorem 1.1, the inclusion $\pi_n^{Sp}(X, x_0) \subseteq \ker(\Psi_n)$ was first proved for $n = 1$ in [18, Prop. 4.8] and for $n \geqslant 2$ in [1, Theorem 4.14]. We include this proof for the sake of completion (Lemma 3.11). The proof of the inclusion $\ker(\Psi_n) \subseteq \pi_n^{Sp}(X, x_0)$ appears in Section 5 and is more intricate, requiring a carefully chosen sequence of open cover refinements using the UV^{n-1} property. These refinements allow one to recursively extend maps on simplicial complexes skeleton-wise. These extension methods, established in Section 4, are similar to methods found in [22, 23].

We also put these extension methods to work in Section 6 where we identify conditions that imply Ψ_n is an isomorphism. In [23], Kozlowski-Segal prove that if X is paracompact Hausdorff and UV^n, then Ψ_n is an isomorphism. In [18], Fischer and Zastrow generalize this result in dimension $n = 1$ by replacing "UV^1" with "locally path connected and semilocally simply connected." Similar, to the approach of Fischer-Zastrow, our use of Spanier groups shows that the existence of *small* null-homotopies of small maps $S^n \to X$ (specifically in dimension n) is not necessary to prove that Ψ_n is injective. We say a space X is *semilocally* π_n*-trivial* if for every $x \in X$ there exists an open neighborhood U of x such that every map $S^n \to U$ is null-homotopic in X. This definition is independent of lower dimensions but certainly $UV^n \Rightarrow (UV^{n-1}$ and semilocally π_n-trivial). Our secondary result is the following.

Theorem 1.2. *Let $n \geqslant 1$ and $x_0 \in X$. If X is paracompact, Hausdorff, UV^{n-1}, and semilocally π_n-trivial, then $\Psi_n : \pi_n(X, x_0) \to \check{\pi}_n(X, x_0)$ is an isomorphism.*

The hypotheses in Theorem 1.2 are the homotopical versions of the hypotheses used in [25] to ensure that the canonical homomorphism $\varphi_* : H_n(X) \to \check{H}_n(X)$ is an isomorphism, see also [10] regarding the surjectivity of φ_*. Although we have only weakened the hypothesis of the Kozlowksi-Segal result in dimension n, Theorem 1.2 formally generalizes the results of both [18] and [22] and does apply to some spaces of interest, namely spaces involving cones over (or attached to) wild spaces (see Examples 7.1 and 7.3). Examples also show that Ψ_n can fail to be an isomorphism if X is semilocally π_n-trivial but not UV^{n-1} (Example 7.4) or if X is UV^{n-1} but not semilocally π_n-trivial (Example 7.5).

2 Preliminaries and Notation

Throughout this paper, X is assumed to be a path-connected topological space with basepoint x_0. The unit interval is denoted I and S^n is the unit n-sphere with basepoint $d_0 = (1, 0, \ldots, 0)$. The n-th homotopy group of (X, x_0) is denoted $\pi_n(X, x_0)$. If $f : (X, x_0) \to (Y, y_0)$ is a based map, then $f_\# : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is the induced homomorphism.

A *path* in a space X is a map $\alpha : I \to X$ from the unit interval. The *reverse* of α is the path given by $\alpha^-(t) = \alpha(1 - t)$ and the concatenation of two paths α, β with $\alpha(1) = \beta(0)$ is denoted $\alpha \cdot \beta$. Similarly, if $f, g : S^n \to X$ are maps based at $x \in X$, then $f \cdot g$ denotes the usual n-loop concatenation and f^- denotes the reverse map. We may write $\prod_{i=1}^{m} f_i$ to denote an m-fold concatenation $f_1 \cdot f_2 \cdot \cdots \cdot f_m$.

2.1 Simplicial complexes

We make heavy use of standard notation and theory of abstract and geometric simplicial complexes, which can be found in texts such as [26] and [28]. We briefly recall relevant notation.

If K is an abstract or geometric simplicial complex and $r \geqslant 0$ is an integer, K_r denotes the r-skeleton of K. If K is abstract, $|K|$ denotes the geometric realization of K. If K is geometric, then $\mathrm{sd}^m K$ denotes the m-th barycentric subdivision of K and if v is a vertex of K, then $\mathrm{st}(v, K)$ denotes the open star of the vertex v. When $L \subseteq K$ is a subcomplex, $\mathrm{sd}^m L$ is a subcomplex of $\mathrm{sd}^m K$. If $\sigma = \{v_0, v_1, \ldots, v_r\}$ is an r-simplex of K, then $[v_0, v_1, ..., v_r]$ denotes the r-simplex of $|K|$ with the indicated orientation.

We frequently make use of the standard n-simplex Δ_n in $\mathbb{R}^n$ spanned by the origin d_0 and standard unit vectors. Since the boundary $\partial\Delta_n = \Delta_n = (\Delta_n)_{n-1}$ is homeomorphic to S^{n-1}, we fix a based homeomorphism $\partial\Delta_n \cong S^{n-1}$ that allows us to represent elements of $\pi_n(X, x_0)$ by maps $(\partial\Delta_{n+1}, d_0) \to (X, x_0)$.

2.2 The Čech expansion and shape homotopy groups

We now recall the construction of the first shape homotopy group $\check{\pi}_1(X, x_0)$ via the Čech expansion. For more details, see [26].

Let $\mathcal{O}(X)$ be the set of open covers of X direct by refinement; we write $\mathscr{U} \preceq \mathscr{V}$ when $\mathscr{V}$ refines $\mathscr{U}$. Similarly, let $\mathcal{O}(X, x_0)$ be the set of open covers with a distinguished element containing the basepoint, i.e. the set of pairs $(\mathscr{U}, U_0)$ where $\mathscr{U} \in \mathcal{O}(X)$, $U_0 \in \mathscr{U}$, and $x_0 \in U_0$. We say $(\mathscr{V}, V_0)$ refines $(\mathscr{U}, U_0)$ if $\mathscr{U} \preceq \mathscr{V}$ and $V_0 \subseteq U_0$.

The nerve of a cover $(\mathscr{U}, U_0) \in \mathcal{O}(X, x_0)$ is the abstract simplicial complex $N(\mathscr{U})$ whose vertex set is $N(\mathscr{U})_0 = \mathscr{U}$ and vertices $A_0, ..., A_n \in \mathscr{U}$ span an n-simplex if $\bigcap_{i=0}^{n} A_i \neq \emptyset$. The vertex U_0 is taken to be the basepoint of the geometric realization $|N(\mathscr{U})|$. Whenever $(\mathscr{V}, V_0)$ refines $(\mathscr{U}, U_0)$, we can construct a simplicial map $p_{\mathscr{U}\mathscr{V}} : N(\mathscr{V}) \to N(\mathscr{U})$, called a *projection*, given by sending a vertex $V \in N(\mathscr{V})$ to a vertex $U \in \mathscr{U}$ such that $V \subseteq U$. In particular, V_0 must be sent to U_0. Any such assignment of vertices extends linearly to a simplicial map. Moreover, the induced map $|p_{\mathscr{U}\mathscr{V}}| : |N(\mathscr{V})| \to |N(\mathscr{U})|$ is unique up to based homotopy. Thus the homomorphism $p_{\mathscr{U}\mathscr{V}\#} : \pi_1(|N(\mathscr{V})|, V_0) \to \pi_1(|N(\mathscr{U})|, U_0)$ induced on fundamental groups is (up to coherent isomorphism) independent of the choice of simplicial map.

Recall that an open cover $\mathscr{U}$ of X is normal if it admits a partition of unity subordinated to $\mathscr{U}$. Let Λ be the subset of $\mathcal{O}(X, x_0)$ (also directed by refinement) consisting of pairs $(\mathscr{U}, U_0)$ where $\mathscr{U}$ is a normal open cover of X and such that there is a partition of unity $\{\phi_U\}_{U \in \mathscr{U}}$ subordinated to $\mathscr{U}$ with $\phi_{U_0}(x_0) = 1$. It is well-known that every open cover of a paracompact Hausdorff space X is normal. Moreover, if $(\mathscr{U}, U_0) \in \mathcal{O}(X, x_0)$, it is easy to refine $(\mathscr{U}, U_0)$ to a cover $(\mathscr{V}, V_0)$ such that V_0 is the only element of $\mathscr{V}$ containing x_0 and therefore $(\mathscr{V}, V_0) \in \Lambda$. Thus, for paracompact Hausdorff X, Λ is cofinal in $\mathcal{O}(X, x_0)$.

The *n-th shape homotopy group* is the inverse limit

$$\check{\pi}_n(X, x_0) = \varprojlim \left(\pi_n(|N(\mathscr{U})|, U_0), p_{\mathscr{U}\mathscr{V}\#}, \Lambda\right).$$

This group is also referred to as the n-th Čech homotopy group.

Given an open cover $(\mathscr{U}, U_0) \in \mathcal{O}(X, x_0)$, a map $p_{\mathscr{U}} : X \to |N(\mathscr{U})|$ is a *(based) canonical map* if $p_{\mathscr{U}}^{-1}(\mathrm{st}(U, N(\mathscr{U}))) \subseteq U$ for each $U \in \mathscr{U}$ and $p_{\mathscr{U}}(x_0) = U_0$. Such a canonical map is guaranteed to exist if $(\mathscr{U}, U_0) \in \Lambda$: find a locally finite partition of unity $\{\phi_U\}_{U \in \mathscr{U}}$ subordinated to $\mathscr{U}$ such that $\phi_{U_0}(x_0) = 1$. When $U \in \mathscr{U}$ and $x \in U$, determine $p_{\mathscr{U}}(x)$ by requiring its barycentric coordinate belonging to the vertex U of $|N(\mathscr{U})|$ to be $\phi_U(x)$. According to this construction, the requirement $\phi_{U_0}(x_0) = 1$ gives $p_{\mathscr{U}}(x_0) = U_0$.

A canonical map $p_{\mathscr{U}}$ is unique up to based homotopy and whenever $(\mathscr{V}, V_0)$ refines $(\mathscr{U}, U_0)$; the compositions $p_{\mathscr{U}\mathscr{V}} \circ p_{\mathscr{V}}$ and $p_{\mathscr{U}}$ are homotopic as based maps. Hence, for $n \geqslant 1$, the homomorphisms $p_{\mathscr{U}\#} : \pi_n(X, x_0) \to \pi_n(|N(\mathscr{U})|, U_0)$ satisfy $p_{\mathscr{U}\mathscr{V}\#} \circ p_{\mathscr{V}\#} = p_{\mathscr{U}\#}$. These homomorphisms induce the following canonical homomorphism to the limit, which is natural in X:

$$\Psi_n : \pi_n(X, x_0) \to \check{\pi}_n(X, x_0) \text{ given by } \Psi_n([f]) = ([p_{\mathscr{U}} \circ f])$$

The subgroup $\ker(\Psi_n)$, which we refer to as the *n-th shape kernel* is, in a sense, a rough algebraic measure of the n-dimensional homotopical information lost when approximating X by polyhedra. Specifically, $[f] \in \pi_n(X, x_0) \backslash \ker(\Psi_n)$ if and only if there exists some polyhedron K and map $p : (X, x_0) \to (K, k_0)$ such that $p_{\#}([f]) \neq 0$ in $\pi_n(K, k_0)$. Of utmost important is the situation when $\ker(\Psi_n) = 1$. In this case, $\pi_n(X, x_0)$ can be understood as a subgroup of $\check{\pi}_n(X, x_0)$, that is, the n-th shape group retains all the data in the n-th homotopy group of X. A space for which $\ker(\Psi_n) = 1$ is said to be π_n-*shape injective.*

3 Higher Spanier Groups

To define higher Spanier groups as in [1], we briefly recall the action of the fundamental groupoid on the higher homotopy groups of a space. Fix a retraction $R : S^n \times I \to S^n \times \{0\} \cup \{d_0\} \times I$. Given a map $f : (S^n, d_0) \to (X, y)$ and a path $\alpha : I \to X$ with $\alpha(0) = x$ and $\alpha(1) = y$, define $F : S^n \times \{0\} \cup \{d_0\} \times I \to X$ so that $g(x, 0) = f(x)$ and $f(d_0, t) = \alpha(1 - t)$. The the *path-conjugate of f by α* is the map $\alpha * f : (S^n, d_0) \to (X, x)$ given by $\alpha * f(x) = F \circ R(x, 0)$.

Path-conjugation defines the basepoint-change isomorphism $\varphi_\alpha : \pi_n(X, y) \to \pi_n(X, x)$, $\varphi_\alpha([f]) = [\alpha * f]$. In particular, $[\alpha * f][\alpha * g] = [\alpha * (f \cdot g)]$ and if $[\alpha] = [\beta]$, then $[\alpha * f] = [\beta * f]$. Note that when $n = 1$, $f : S^1 \to X$ is a loop and $\alpha * f \simeq \alpha \cdot f \cdot \alpha^-$.

Definition 3.1. Let $n \geqslant 1$ and $\alpha : (I, 0) \to (X, x_0)$ be a path and U be an open neighborhood of $\alpha(1)$ in X. Define

$$[\alpha] * \pi_n(U) = \{[\alpha * f] \in \pi_n(X, x_0) \mid f(S^n) \subseteq U\}.$$

Since $[\alpha * f][\alpha * g] = [\alpha * (f \cdot g)]$, the set $[\alpha] * \pi_n(U)$ is a subgroup of $\pi_n(X, x_0)$.

Definition 3.2. Let $n \geqslant 1$, $\mathscr{U}$ be an open cover of X, and $x_0 \in X$. The *n-th Spanier group of (X, x_0) with respect to $\mathscr{U}$* is the subgroup $\pi_n^{Sp}(\mathscr{U}, x_0)$ of $\pi_n(X, x_0)$ generated by the subgroups $[\alpha] * \pi_n(U)$ for all pairs (α, U) with $\alpha(1) \in U$ and $U \in \mathscr{U}$. In short:

$$\pi_n^{Sp}(\mathscr{U}, x_0) = \langle [\alpha] * \pi_n(U) \mid U \in \mathscr{U}, \alpha(1) \in U \rangle$$

The *n-th Spanier group of* (X, x_0) is the intersection

$$\pi_n^{Sp}(X, x_0) = \bigcap_{\mathscr{U} \in O(X)} \pi_n^{Sp}(\mathscr{U}, x_0).$$

Remark 3.3. We note that our definition of n-th Spanier group is the "unbased" definition from [1]; see also [16] for more on "based" Spanier groups, which is defined using covers of X by *pointed* open sets. The two notions agree for locally path connected spaces. When $n = 1$, Spanier groups (absolute and relative to a cover) are normal subgroups of $\pi_1(X, x_0)$. Certainly, the same is true for $n \geqslant 2$ since higher homotopy groups are abelian. In the case $n = 1$, Spanier groups have been studied heavily due to their relationship to covering space theory [30].

Remark 3.4 (Functorality)**.** If $f : (X, x_0) \to (Y, y_0)$ is a map and $\mathscr{V}$ is an open cover of Y, then $\mathscr{U} = \{f^{-1}(V) \mid V \in \mathscr{V}\}$ is an open cover of X such that $f_\#(\pi_n(\mathscr{U}, x_0)) \subseteq \pi_n(\mathscr{V}, y_0)$. It follows that $f_\#(\pi_n^{Sp}(X, x_0)) \subseteq \pi_n^{Sp}(Y, y_0)$. Thus $(f_\#)|_{\pi_n^{Sp}(X,x_0)} : \pi_n^{Sp}(X, x_0) \to \pi_n^{Sp}(Y, y_0)$ is well-defined showing that $\pi_1^{Sp} : \mathbf{Top}_* \to \mathbf{Grp}$ and $\pi_n^{Sp} : \mathbf{Top}_* \to \mathbf{Ab}$, $n \geqslant 2$, are functors [1, Theorem 4.2]. Moreover, if $g : (Y, y_0) \to (X, x_0)$ is a based homotopy inverse of f, then $(f_\#)|_{\pi_n^{Sp}(X,x_0)}$ and $(g_\#)|_{\pi_n^{Sp}(Y,y_0)}$ are inverse isomorphisms. Hence, these functors descend to functors $\mathbf{hTop}_* \to \mathbf{Grp}$ and $\mathbf{hTop}_* \to \mathbf{Ab}$ on the based homotopy category.

Remark 3.5 (Basepoint invariance)**.** Suppose $x_0, x_1 \in X$ and $\beta : I \to X$ is a path from x_1 to x_0, and $\varphi_\beta : \pi_n(X, x_0) \to \pi_n(X, x_1)$, $\varphi_\beta([g]) = [\beta * g]$ is the basepoint-change isomorphism. If $[\alpha * f]$ is a generator of $\pi_n^{Sp}(\mathscr{U}, x_0)$, then $\varphi_\beta([\alpha * f]) = [(\beta \cdot \alpha) * f]$ is a generator of $\pi_n^{Sp}(\mathscr{U}, x_1)$. It follows that $\varphi_\beta(\pi_n^{Sp}(\mathscr{U}, x_0)) = \pi_n^{Sp}(\mathscr{U}, x_1)$. Moreover, in the absolute case, we have $\varphi_\beta(\pi_n^{Sp}(X, x_0)) = \pi_n^{Sp}(X, x_1)$. In particular, changing the basepoint of X does not change the isomorphism type of the n-th Spanier group, particularly whether it is trivial or not.

In terms of our choice of generators, a generic element of $\pi_n^{Sp}(\mathscr{U}, x_0)$ is a product $\prod_{i=1}^{m} [\alpha_i * f_i]$ where each map $f_i : S^n \to X$ has an image in some open set $U_i \in \mathscr{U}$ (see Figure 1). The next lemma identifies how such products might actually appear in practice and motivates the proof of our key technical Lemma below (Lemma 5.1). Recall that $(\mathrm{sd}^m \Delta_{n+1})_n$ is the union of the boundaries of the $(n+1)$-simplices in the m-th barycentric subdivision $\mathrm{sd}^m \Delta_{n+1}$.

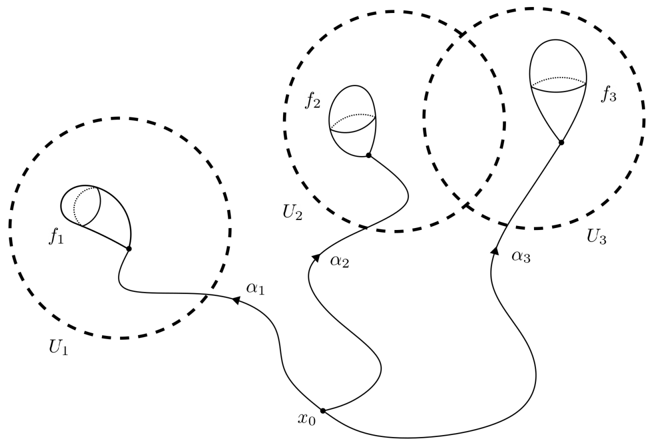

Figure 1: An element of $\pi_2^{Sp}(\mathscr{U}, x_0)$, which is a product of three path-conjugate generators $[\alpha_i * f_i]$.

Lemma 3.6. *If $m, n \in \mathbb{N}$, $\mathscr{U}$ is an open cover of X, and $f : ((sd^m\Delta_{n+1})_n, d_0) \to (X, x_0)$ is a map such that for every $(n+1)$-simplex σ of $sd^m\Delta_{n+1}$, we have $f(\partial\sigma) \subseteq U$ for some $U \in \mathscr{U}$, then $f_\#(\pi_n((sd^m\Delta_{n+1})_n, d_0)) \subseteq \pi_n^{Sp}(\mathscr{U}, x_0)$.*

Proof. The case $n = 1$ is proved in [4]. Suppose $n \geqslant 2$ and set $K = \text{sd}^m\Delta_{n+1}$. The set $\mathscr{W} = \{f^{-1}(U) \mid U \in \mathscr{U}\}$ is an open cover of K_n such that $f_\#(\pi_n^{Sp}(\mathscr{W}, d_0)) \subseteq \pi_n^{Sp}(\mathscr{U}, x_0)$ and for every $(n+1)$-simplex σ in K, we have $\partial\sigma \subseteq f^{-1}(U)$ for some $U \in \mathscr{U}$. Thus it suffices to prove $\pi_n^{Sp}(\mathscr{W}, d_0) = \pi_n(K_n, d_0)$. Let S be the set of n-simplices of K. Since $n \geqslant 2$, K_n is simply connected. Standard simplicial homology arguments give that the reduced singular homology groups of K_n are trivial in dimension $< n$ and $H_n(K_n)$ is finitely generated free abelian generated. A set of free generators for $H_n(K_n)$ can be chosen by fixing the homology class of a simplicial map $g_\sigma : \partial\Delta_{n+1} \to K_n$ that sends $\partial\Delta_{n+1}$ homeomorphically onto the boundary of an $(n+1)$-simplex of $\sigma \in S$. Thus K_n is $(n-1)$-connected and the Hurewicz homomorphism $h : \pi_k(K_n, d_0) \to H_k(K_n)$ is an isomorphism for all $1 \leqslant k \leqslant n$. In particular, let $p_\sigma : I \to K_n$ be any path from d_0 to $g_\sigma(d_0)$. Then $\pi_n(K_n, d_0)$ is freely generated by the path-conjugates $[p_\sigma * g_\sigma]$, $\sigma \in S$. By assumption, for every $\sigma \in S$, $[p_\sigma * g_\sigma]$ is a generator of $\pi_n^{Sp}(\mathscr{W}, d_0)$. Since $\pi_n^{Sp}(\mathscr{W}, d_0)$ contains all the generators of $\pi_n(K_n, d_0)$, the equality $\pi_n^{Sp}(\mathscr{W}, d_0) = \pi_n(K_n, d_0)$ follows. □

To characterize the triviality of relative Spanier groups, we establish the following terminology.

Definition 3.7. Let $n \geqslant 0$. We say a space X is

(1) *semilocally π_n-trivial at $x \in X$* if there exists an open neighborhood U of X such that every map $S^n \to U$ is null-homotopic in X.

(2) *semilocally n-connected at $x \in X$* if there exists an open neighborhood U of X such that every map $S^k \to X$, $0 \leqslant k \leqslant n$ is null-homotopic in X.

We say X is *semilocally π_n-trivial* (resp. semilocally n-connected) if it has this property at all of its points.

It is straightforward to see that X is semilocally n-connected at $x \in X$ if and only if X is semilocally π_k-trivial for all $0 \leqslant k \leqslant n$.

Remark 3.8. Note that a space X is semilocally π_n-trivial if and only if X admits an open cover $\mathscr{U}$ such that $\pi_n^{Sp}(\mathscr{U}, x_0)$ is trivial [1, Theorem 3.7]. Moreover, X is semilocally n-connected if and only if X admits an open cover $\mathscr{U}$ such that $\pi_k^{Sp}(\mathscr{U}, x_0)$ is trivial for all $1 \leqslant k \leqslant n$.

Attempting a proof of Theorem 1.1, one should not expect the groups $\pi_n^{Sp}(\mathscr{U}, x_0)$ and $\ker(p_{\mathscr{U}\#})$ to agree "on the nose." Indeed, the following example shows that we should not expect the equality $\pi_n^{Sp}(\mathscr{U}, x_0) = \ker(p_{\mathscr{U}\#})$ to hold even in the "nicest" local circumstances.

Example 3.9. Let $X = S^2 \vee S^2$ and W be a contractible neighborhood of d_0 in S^2. Set $U_1 = S^2 \vee W$ and $U_2 = W \vee S^2$ and consider the open cover $\mathscr{U} = \{U_1, U_2\}$ of X. Then $\pi_3^{Sp}(\mathscr{U}, x_0) \cong \mathbb{Z}^2$ is freely generated by the homotopy classes of the two inclusions $i_1, i_2 : S^2 \to X$. However, $\pi_3(X) \cong \mathbb{Z}^3$ is freely generated by $[i_1]$, $[i_2]$, and the Whitehead product $[\![i_1, i_2]\!]$. However $|N(\mathscr{U})|$ is a 1-simplex and is therefore contractible. Thus $\ker(p_{\mathscr{U}\#})$ is equal to $\pi_3(X)$ and contains $[\![i_1, i_2]\!]$. Even though the spaces X, U_1, U_2 are locally contractible and the elements of $\mathscr{U}$ are 1-connected, $\pi_n^{Sp}(\mathscr{U}, x_0)$ is a proper subgroup of $\ker(p_{\mathscr{U}\#})$. One can view this failure as the result of two facts: (1) The sets U_i are not 2-connected and (2) the definition of Spanier group does not allow one to generate homotopy classes by taking Whitehead products of maps $S^2 \to U_i$ in the neighboring elements of $\mathscr{U}$.

First, we show the inclusion $\pi_n^{Sp}(X, x_0) \subseteq \ker(\Psi_n)$ holds in full generality. Recall the intersections $\pi_n^{Sp}(X, x_0) = \bigcap_{\mathscr{U} \in O(X)} \pi_n^{Sp}(\mathscr{U}, x_0)$ and $\ker(\Psi_n) = \bigcap_{(\mathscr{U}, U_0) \in \Lambda} \ker(p_{\mathscr{U}\#})$ are formally indexed by different sets.

Lemma 3.10. *For every open cover $\mathscr{U}$ of X and canonical map $p_{\mathscr{U}} : X \to |N(\mathscr{U})|$, there exists a refinement $\mathscr{U} \preceq \mathscr{V}$ such that $\pi_n^{Sp}(\mathscr{V}, x_0) \subseteq \ker(p_{\mathscr{U}\#})$ in $\pi_n(X, x_0)$.*

Proof. Let $\mathscr{U} \in O(X)$. The stars $\mathrm{st}(U, |N(\mathscr{U})|)$, $U \in \mathscr{U}$ form an open cover of $|N(\mathscr{U})|$ and therefore $\mathscr{V} = \{p_{\mathscr{U}}^{-1}(\mathrm{st}(U, |N(\mathscr{U})|)) \mid U \in \mathscr{U}\}$ is an open cover of X. Since $p_{\mathscr{U}}$ is a canonical map, we have $p_{\mathscr{U}}^{-1}(\mathrm{st}(U, |N(\mathscr{U})|)) \subseteq U$ for all $U \in \mathscr{U}$. Thus $\mathscr{V}$ is a refinement of $\mathscr{U}$. A generator of $\pi_n^{Sp}(\mathscr{V}, x_0)$ is of the form $[\alpha * f]$ for a map $f : S^n \to p_{\mathscr{U}}^{-1}(\mathrm{st}(U, |N(\mathscr{U})|))$. However, $p_{\mathscr{U}} \circ f$ has image in the contractible open set $\mathrm{st}(U, |N(\mathscr{U})|)$ and is therefore null-homotopic. Thus $p_{\mathscr{U}\#}([\alpha * f]) = 0$. We conclude that $p_{\mathscr{U}\#}(\pi_n^{Sp}(\mathscr{V}, x_0)) = 0$. □

Corollary 3.11. *[1, Theorem 4.14] Let $n \geqslant 1$. For any based space (X, x_0), we have $\pi_n^{Sp}(X, x_0) \subseteq \ker(\Psi_n)$.*

Proof. Suppose $[f] \in \pi_n^{Sp}(X, x_0)$. Given a normal, based open cover $(\mathscr{U}, U_0) \in \Lambda$ and any canonical map $p_{\mathscr{U}} : X \to |N(\mathscr{U})|$, Lemma 3.10 ensures we can find a refinement $\mathscr{U} \preceq \mathscr{V}$ such that $\pi_n^{Sp}(\mathscr{V}, x_0) \subseteq \ker(p_{\mathscr{U}\#})$. Thus $[f] \in \pi_n^{Sp}(\mathscr{V}, x_0) \subseteq \ker(p_{\mathscr{U}\#})$, which shows that $[f] \in \ker(\Psi_n)$. □

Example 3.12 (higher earring spaces)**.** An important space, which we will call upon repeatedly for examples, is the *n-dimensional earring space*

$$\mathbb{E}_n = \bigcup_{j \in \mathbb{N}} \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x} - (1/j, 0, 0, \ldots, 0)\| = 1/j \right\},$$

which is a shrinking wedge (one-point union) of n-spheres with basepoint $b_0 = (0, 0, \ldots, 0)$. It is known that $\mathbb{E}_n$ is $(n-1)$-connected, locally $(n-1)$-connected, and π_n-shape injective for all $n \geqslant 1$ [27, 12]. However, $\mathbb{E}_n$ is not semilocally π_n-trivial. Thus $\pi_n^{Sp}(\mathscr{U}, b_0) \neq 0$ for any open cover $\mathscr{U}$ of $\mathbb{E}_n$ even though "in the limit" $\pi_n^{Sp}(\mathbb{E}_n, b_0)$ is trivial.

Example 3.13. Let $n \geqslant 3$ and notice that $\mathbb{E}_1 \vee \mathbb{E}_n$ is not semilocally π_1-connected (since it has $\mathbb{E}_1$ as a retract) and therefore fails to be semilocally $(n-1)$-connected. However, it has recently been shown that $\pi_k(\mathbb{E}_1 \vee \mathbb{E}_n) = 0$ for $2 \leqslant k \leqslant n-1$ and that $\mathbb{E}_1 \vee \mathbb{E}_n$ is π_n-shape injective [3]. Thus $\mathbb{E}_1 \vee \mathbb{E}_n$ is semilocally π_k-trivial for all $k \leqslant n-1$ except $k = 1$ and $\pi_n^{Sp}(\mathbb{E}_1 \vee \mathbb{E}_n, b_0) = 0$. Thus the failure to be semilocally n-connected can occur at single dimension less than n.

4 Recursive Extension Lemmas

Toward a proof of the inclusion $\ker(\Psi_n) \subseteq \pi_n^{Sp}(X, x_0)$, we introduce some convenient notation and definitions. If $\mathscr{U}$ is an open cover and $A \subseteq X$, then $\mathrm{St}(A, \mathscr{U}) = \bigcup\{U \in \mathscr{U} \mid A \cap U \neq \varnothing\}$. Note that if $A \subseteq B$, then $\mathrm{St}(A, \mathscr{U}) \subseteq \mathrm{St}(B, \mathscr{U})$. Also if $\mathscr{U} \preceq \mathscr{V}$, then $\mathrm{St}(A, \mathscr{V}) \subseteq \mathrm{St}(A, \mathscr{U})$. We take the following terminology from [33].

Definition 4.1. Let $\mathscr{U}, \mathscr{V} \in O(X)$.

(1) We say $\mathscr{V}$ is a *barycentric-star refinement* of $\mathscr{U}$ if for every $x \in X$, we have $\mathrm{St}(x, \mathscr{V}) \subseteq U$ for some $U \in \mathscr{U}$. We write $\mathscr{U} \preceq_* \mathscr{V}$.

(2) We say $\mathscr{V}$ is a *star refinement* of $\mathscr{U}$ if for every $V \in \mathscr{V}$, we have $\mathrm{St}(V, \mathscr{V}) \subseteq U$ for some $U \in \mathscr{U}$. We write $\mathscr{U} \preceq_{**} \mathscr{V}$.

Note that if $\mathscr{U} \preceq_* \mathscr{V} \preceq_* \mathscr{W}$, then $\mathscr{U} \preceq_{**} \mathscr{W}$.

Lemma 4.2. *[31] A T_1 space X is paracompact if and only if for every open cover $\mathscr{U}$ of X there exists an open cover $\mathscr{V}$ such that $\mathscr{U} \preceq_* \mathscr{V}$.*

Definition 4.3. [29] Let $n \in \{0, 1, 2, 3, \ldots, \infty\}$. A space X is UV^n at $x \in X$ and every neighborhood U of x, there exists a neighborhood V of x such that $V \subseteq U$ and such that for all $0 \leqslant k \leqslant n$ ($k < \infty$ if $n = \infty$), every map $f : \partial\Delta_{k+1} \to V$ extends to a map $g : \Delta_{k+1} \to U$. We say X is UV^n if X is UV^n at all of its points.

We have the following evident implications for both the point-wise and global properties:

$$X \text{ is locally } n\text{-connected} \;\Rightarrow\; X \text{ is } UV^n \;\Rightarrow\; X \text{ is semilocally } n\text{-connected}$$

For first countable spaces, the UV^n property is equivalent to the "n-tame" property in [3] defined in terms of shrinking sequences of maps.

Remark 4.4. In much of the Shape Theory literature, the UV^n property is referred to as the "LC^n property" [26, p. 40]. This is sometimes confused with local n-connectedness in which one has a basis of n-connected open sets. Since the two are not equivalent even for Peano continua, we prefer the "UV^n" terminology.

Definition 4.5. Suppose $\mathscr{U} \preceq \mathscr{V}$ in $O(X)$.

(1) We say $\mathscr{V}$ is an *n-refinement of* $\mathscr{U}$, and write $\mathscr{U} \preceq^n \mathscr{V}$, if for all $1 \leqslant k \leqslant n$, $V \in \mathscr{V}$, and maps $f : \partial\Delta_{k+1} \to V$, there exists $U \in \mathscr{U}$ with $V \subseteq U$ and a continuous extension $g : \Delta_{k+1} \to U$ of f.

(2) We say $\mathscr{V}$ is an *n-barycentric-star refinement of* $\mathscr{U}$, and write $\mathscr{U} \preceq^n_* \mathscr{V}$, if for every $0 \leqslant k \leqslant n$, for every $x \in X$, and every map $f : \partial\Delta_{k+1} \to \mathrm{St}(x, \mathscr{V})$, there exists $U \in \mathscr{U}$ with $\mathrm{St}(x, \mathscr{V}) \subseteq U$ and a continuous extension $g : \Delta_{k+1} \to U$ of f.

Note that if $\mathscr{U} \preceq^n \mathscr{V}$ (resp. $\mathscr{U} \preceq^n_* \mathscr{V}$), then $\mathscr{U} \preceq^k \mathscr{V}$ (resp. $\mathscr{U} \preceq^k_* \mathscr{V}$) for all $0 \leqslant k \leqslant n$.

Lemma 4.6. *Suppose X is paracompact, Hausdorff, and UV^n. For every $\mathscr{U} \in O(X)$, there exists $\mathscr{V} \in O(X)$ such that $\mathscr{U} \preceq^n_* \mathscr{V}$.*

Proof. Let $\mathscr{U} \in O(X)$. Since X is UV^n, for every $U \in \mathscr{U}$ and $x \in U$, there exists an open neighborhood $W(U, x)$ such that $W(U, x) \subseteq U$ and such that for all $1 \leqslant k \leqslant n$, each map $f : \partial\Delta_{k+1} \to W(U, x)$ extends to a map $g : \Delta_{k+1} \to U$. Let $\mathscr{W} = \{W(U, x) \mid U \in \mathscr{U}, x \in U\}$ and note $\mathscr{U} \preceq^n \mathscr{W}$. Since X is paracompact Hausdorff, by Lemma 4.2, there exists $\mathscr{V} \in O(X)$ such that $\mathscr{W} \preceq_* \mathscr{V}$.

Fix $x' \in X$. Then $\mathrm{St}(x', \mathscr{V}) \subseteq W(U, x)$ for some $x \in U \in \mathscr{U}$. Then $\mathrm{St}(x', \mathscr{V}) \subseteq U$. Moreover, if $1 \leqslant k \leqslant n$ and $f : \partial\Delta_{k+1} \to \mathrm{St}(x', \mathscr{V})$ is a map, then since f has image in $W(U, x)$, there is an extension $g : \Delta_{k+1} \to U$. This verifies that $\mathscr{U} \preceq^n_* \mathscr{V}$. □

For the next two lemmas, we fix $n \in \mathbb{N}$, a geometric simplicial complex K consisting of $(n+1)$-simplices and their faces, and a subcomplex $L \subseteq K$ with $\dim(L) \leqslant n$. Let $M[k] = L \cup K_k$ denote the union of L and the k-skeleton of K. Since $L \subseteq K_n$, $M[n] = K_n$ is the union of the boundaries of the $(n+1)$-simplices of K. Later we will consider the cases where (1) $K = \mathrm{sd}^m \Delta_{n+1}$ and $L = \mathrm{sd}^m \partial\Delta_{n+1}$ and (2) $K = \mathrm{sd}^m \partial\Delta_{n+2}$ and $L = \{d_0\}$.

Lemma 4.7 (Recursive Extensions). *Suppose $1 \leqslant k \leqslant n$, $\mathscr{U} \preceq_* \mathscr{V} \preceq^{k-1}_* \mathscr{W}$, $m \in \mathbb{N}$, and $f : M[k-1] \to X$ is a map such that for every $n+1$-simplex σ of K, we have $f(\sigma \cap M[k-1]) \subseteq W_\sigma$ for some $W_\sigma \in \mathscr{W}$. Then there exists a continuous extension $g : M[k] \to X$ of f such that for every $(n+1)$-simplex σ of K, we have $g(\sigma \cap M[k]) \subseteq U_\sigma$ for some $U_\sigma \in \mathscr{U}$.*

Proof. Supposing the hypothesis, we must extend f to the k-simplices of $M[k]$ that do not lie in L. Let τ be a k-simplex of $M[k]$ that does not lie in L and let S_τ be the set of $(n+1)$-simplices in K that contain τ. By assumption, S_τ is non-empty. We make some general observations first. Since f maps the $(k-1)$-skeleton of each $(n+1)$-simplex $\sigma \in S_\tau$ into W_σ and $\partial\tau$ lies in this $(k-1)$-skeleton, we have $f(\partial\tau) \subseteq \bigcap_{\sigma \in S_\tau} W_\sigma$. Thus, for all τ, we have

$$f(\partial\tau) \subseteq \bigcap_{\sigma \in S_\tau} \mathrm{St}(W_\sigma, \mathscr{V}).$$

Fix a vertex v_τ of τ and let $x_\tau = f(v_\tau)$. Then $x_\tau \in W_\sigma \subseteq \mathrm{St}(x_\tau, \mathscr{W})$ whenever $\sigma \in S_\tau$. Since $\mathscr{V} \preceq_*^{k-1} \mathscr{W}$, we may find $V_\tau \in \mathscr{V}$ such that $\mathrm{St}(x_\tau, \mathscr{W}) \subseteq V_\tau$ and such that every map $\partial\Delta_k \to \mathrm{St}(x_\tau, \mathscr{W})$ extends to a map $\Delta_k \to V_\tau$. In particular, $f|_{\partial\tau} : \partial\tau \to W_\sigma$ extends to a map $\tau \to V_\tau$. We define $g : M[k] \to X$ so that it agrees with f on $M[k-1]$ and so that the restriction of g to τ is a choice of continuous extension $\tau \to V_\tau$ of $f|_{\partial\tau}$.

We now choose the sets U_σ. Fix an $(n+1)$-simplex σ of K. If the k-skeleton of σ lies entirely in L, we choose any $U_\sigma \in \mathscr{U}$ satisfying $W_\sigma \subseteq U_\sigma$. Suppose there exists at least one k-simplex in σ not in L. Then whenever τ is a k-simplex of σ not in L, we have $W_\sigma \subseteq \mathrm{St}(x_\tau, \mathscr{W}) \subseteq V_\tau$. Fix a point $y_\sigma \in W_\sigma$. The assumption that $\mathscr{U} \preceq_* \mathscr{V}$ implies that there exists $U_\sigma \in \mathscr{U}$ such that $\mathrm{St}(y_\sigma, \mathscr{V}) \subseteq U_\sigma$. In this case, we have $W_\sigma \subseteq V_\tau \subseteq U_\sigma$ whenever τ is a k-simplex of σ not in L.

Finally, we check that g satisfies the desired property. Again, fix an $(n+1)$-simplex σ of K. If τ is a k-simplex of σ not in L, our definition of g gives $g(\tau) \subseteq V_\tau \subseteq U_\sigma$. If τ' is a k-simplex in $\sigma \cap L$, then $g(\tau') = f(\tau') \subseteq W_\sigma \subseteq U_\sigma$. Overall, this shows that $g(\sigma \cap M[k]) \subseteq U_\sigma$ for each $(n+1)$-simplex σ of K. □

A direct, recursive application of the previous lemma is given in the following statement.

Lemma 4.8. *Suppose there is a sequence of open covers*

$$\mathscr{U} = \mathscr{W}_n \preceq_* \mathscr{V}_n \preceq_*^{n-1} \mathscr{W}_{n-1} \preceq_* \cdots \preceq_*^2 \mathscr{W}_2 \preceq_* \mathscr{V}_2 \preceq_*^1 \mathscr{W}_1 \preceq_* \mathscr{V}_1 \preceq_*^0 \mathscr{W}_0 = \mathscr{W}$$

and a map $f_0 : M[0] \to X$ such that for every $(n+1)$-simplex σ of K, we have $f_0(\sigma \cap M[0]) \subseteq W$ for some $W \in \mathscr{W}$. Then there exists an extension $f_n : M[n] \to X$ of f_0 such that for every $(n+1)$-simplex σ of K, we have $f_n(\partial\sigma) \subseteq U$ for some $U \in \mathscr{U}$.

5 A proof of Theorem 1.1

We apply the extension results of the previous section in the case where $K = \mathrm{sd}^m \Delta_{n+1}$ for some $m \in \mathbb{N}$ and $L = \mathrm{sd}^m \partial\Delta_{n+1}$ so that $M[k] = L \cup K_k$ consists of the boundary of Δ_{n+1} and the k-simplices of $\mathrm{sd}^m \Delta_{n+1}$ not in the boundary. Note that $M[n]$ is the union of the boundaries of the $(n+1)$-simplices of $\mathrm{sd}^m \Delta_{n+1}$.

Lemma 5.1. *Let $n \geqslant 1$. Suppose X is paracompact, Hausdorff, and UV^{n-1}. Then for every open cover $\mathscr{U}$ of X, there exists $(\mathscr{V}, V_0) \in \Lambda$ such that* $\ker(p_{\mathscr{V}\#}) \subseteq \pi_n^{Sp}(\mathscr{U}, x_0)$.

Proof. Suppose $\mathscr{U} \in O(X)$. Since X is paracompact, Hausdorff, and UV^{n-1}, we may apply Lemmas 4.2 and 4.6 to first find a sequence of refinements :

$$\mathscr{U} = \mathscr{U}_n \preceq_* \mathscr{V}_n \preceq_*^{n-1} \mathscr{U}_{n-1} \preceq_* \cdots \preceq_*^2 \mathscr{U}_2 \preceq_* \mathscr{V}_2 \preceq_*^1 \mathscr{U}_1 \preceq_* \mathscr{V}_1 \preceq_*^0 \mathscr{U}_0$$

and then one last refinement $\mathscr{U}_0 \preceq_* \mathscr{V}_0 = \mathscr{V}$. Let $V_0 \in \mathscr{V}$ be any set containing x_0 and recall that since X is paracompact Hausdorff $(\mathscr{V}, V_0) \in \Lambda$. We will show that $\ker(p_{\mathscr{V}\#}) \subseteq \pi_n^{Sp}(\mathscr{U}, x_0)$. Note that $p_{\mathscr{V}}^{-1}(\mathrm{st}(V, N(\mathscr{V}))) \subseteq V$ for some choice of canonical map $p_{\mathscr{V}}$.

Suppose $[f] \in \ker(p_{\mathscr{V}\#})$ is represented by a map $f : (|\partial\Delta_{n+1}|, d_0) \to (X, x_0)$. We will show that $[f] \in \pi_n^{Sp}(\mathscr{U}, x_0)$. Then $p_{\mathscr{V}} \circ f : |\partial\Delta_{n+1}| \to |N(\mathscr{V})|$ is null-homotopic and extends to a map $h : |\Delta_{n+1}| \to |N(\mathscr{V})|$. Set $Y_V = h^{-1}(\mathrm{st}(V, N(\mathscr{V})))$ so that $\mathscr{Y} = \{Y_V \mid V \in \mathscr{V}\}$ is an open cover of $|\Delta_{n+1}|$.

We find a particular simplicial approximation for h using the cover $\mathscr{Y}$ [28, Theorem 16.1]: let λ be a Lebesgue number for $\mathscr{Y}$ so that any subset of Δ_{n+1} of diameter less than λ lies in some element of $\mathscr{Y}$. Find $m \in \mathbb{N}$ such that each simplex in $\mathrm{sd}^m\Delta_{n+1}$ has diameter less than $\lambda/2$. Thus the star $\mathrm{st}(a, \mathrm{sd}^m\Delta_{n+1})$ of each vertex a in $\mathrm{sd}^m\Delta_{n+1}$ lies in a set $Y_{V_a} \in \mathscr{Y}$ for some $V_a \in \mathscr{V}$. The assignment $a \mapsto V_a$ on vertices extends to a simplicial approximation $h' : \mathrm{sd}^m\Delta_{n+1} \to N(\mathscr{V})$ of h, i.e. a simplicial map h' such that

$$h(\mathrm{st}(a, \mathrm{sd}^m\Delta_{n+1})) \subseteq \mathrm{st}(h'(a), N(\mathscr{V})) = \mathrm{st}(V_a, N(\mathscr{V}))$$

for each vertex a [28, Lemma 14.1].

Let $K = \mathrm{sd}^m\Delta_{n+1}$ and $L = \mathrm{sd}^m\partial\Delta_{n+1}$ so that $M[k] = L \cup K_k$. First, we extend $f : L \to X$ to a map $f_0 : M[0] \to X$. For each vertex a in K, pick a point $f_0(a) \in V_a$. In particular, if $a \in L$, take $f_0(a) = f(a)$. This choice is well defined since on boundary vertices $a \in L$ since we have $p_{\mathscr{V}} \circ f(a) = h(a) \in \mathrm{st}(V_a, |N(\mathscr{V})|)$ and thus $f(a) \in p_{\mathscr{V}}^{-1}(\mathrm{st}(V_a, |N(\mathscr{V}|))) \subseteq V_a$.

Note that h' maps every simplex $\sigma = [a_0, a_1, \ldots, a_k]$ of K to the simplex of $N(\mathscr{V})$ spanned by $\{h'(a_i) \mid 0 \leqslant i \leqslant k\} = \{V_{a_i} \mid 0 \leqslant i \leqslant k\}$. By definition of the nerve, we have $\bigcap\{V_{a_i} \mid 0 \leqslant i \leqslant k\} \neq \varnothing$. Pick a point $x_\sigma \in \bigcap\{V_{a_i} \mid 0 \leqslant i \leqslant k\}$.

By our initial choice of refinements, we have $\mathscr{U}_0 \preceq_* \mathscr{V}$. If $\sigma = [a_0, a_1, \ldots, a_{n+1}]$ is an $(n+1)$-simplex of K, then $\mathrm{St}(x_\sigma, \mathscr{V}) \subseteq U_\sigma$ for some $U_\sigma \in \mathscr{U}$. In particular $\{f_0(a_i) \mid 0 \leqslant i \leqslant n+1\} \subseteq \bigcup\{V_{a_i} \mid 0 \leqslant i \leqslant n+1\} \subseteq U_\sigma$. Thus f_0 maps the 0-skeleton of σ into U_σ. If $1 \leqslant k \leqslant n$, τ is a k-face of $\sigma \cap L$ with $a_i \in \tau$, then $p_{\mathscr{V}} \circ f_0(\mathrm{int}(\tau)) = p_{\mathscr{V}} \circ f(\mathrm{int}(\tau)) = h(\mathrm{int}(\tau)) \subseteq h(\mathrm{st}(a_i, K)) \subseteq \mathrm{st}(V_{a_i}, |N(\mathscr{V})|)$. It follows that

$$f_0(\tau) \subseteq p_{\mathscr{V}}^{-1}(\mathrm{st}(V_{a_i}, |N(\mathscr{V})|)) \subseteq V_{a_i} \subseteq U_\sigma.$$

Thus for every n-simplex in $\sigma \cap L$, we have $f_0(\tau) \subseteq U_\sigma$. We conclude that for every $(n+1)$-simplex σ of K, we have $f_0(\sigma \cap M[0]) \subseteq U_\sigma$.

By our choice of sequence of refinements, we are precisely in the situation to apply Lemma 4.8. Doing so, we obtain an extension $f_n : M[n] \to X$ of f

such that for every $(n+1)$-simplex σ of K, we have $f_n(\partial\sigma) \subseteq \mathbf{U}_\sigma$ for some $\mathbf{U}_\sigma \in \mathscr{U}_n = \mathscr{U}$. By Lemma 3.6, we have $[f] = [f_n|_{\partial\Delta_{n+1}}] \in \pi_n^{Sp}(\mathscr{U}, x_0)$. ☐

Finally, both inclusions have been established and provide a proof of our main result.

Proof of Theorem 1.1. The inclusion $\pi_n^{Sp}(X, x_0) \subseteq \ker(\Psi_n)$ holds in general by Corollary 3.11. Under the given hypotheses, the inclusion $\ker(\Psi_n) \subseteq \pi_n^{Sp}(X, x_0)$ follows from Lemma 5.1. ☐

When considering examples relevant to Theorem 1.1, it is helpful to compare π_n-shape injectivity with the following weaker property from [19].

Definition 5.2. We say a space X is *n-homotopically Hausdorff at $x \in X$* if no non-trivial element of $\pi_n(X, x)$ has a representing map in every neighborhood of x. We say X is *n-homotopically Hausdorff* if it is n-homotopically Hausdorff at all of its points.

Clearly, π_n-shape injectivity $\Rightarrow$ n-homotopically Hausdorff. The next example, which highlights the effectiveness of Theorem 1.1, shows the converse is not true even for UV^{n-1} Peano continua.

Example 5.3. Fix $n \geqslant 2$ and let $\ell_j : S^n \to \mathbb{E}_n$ be the inclusion of the j-th sphere and define $f : \mathbb{E}_n \to \mathbb{E}_n$ to be the shift map given by $f \circ \ell_j = \ell_{j+1}$. Let $M_f = \mathbb{E}_n \times [0,1]/\sim$, $(x, 0) \sim (f(x), 1)$ be the mapping torus of f. We identify $\mathbb{E}_n$ with the image of $\mathbb{E}_n \times \{0\}$ in M_f and take b_0 to be the basepoint of M_f. Let $\alpha : I \to M_f$ be the loop where $\alpha(t)$ is the image of (b_0, t). Then M_f is locally contractible at all points other than those in the image of α. Also, every point $\alpha(t)$ has a neighborhood that deformation retracts onto a homeomorphic copy of $\mathbb{E}_n$. Thus, since $\mathbb{E}_n$ is UV^{n-1}, so is X. It follows from Theorem 1.1 that $\pi_n^{Sp}(M_f, b_0) = \ker(\pi_n(M_f, b_0) \to \check{\pi}_n(M_f, b_0))$. In particular, the Spanier group of M_f contains all elements $[\alpha^k * g]$ where $g : S^n \to \mathbb{E}_n$ is a based map and $k \in \mathbb{Z}$. Using the universal covering map $E \to M_f$ that "unwinds" α and the relation $[g] = [\alpha * (f \circ g)]$ in $\pi_n(M_f, b_0)$, it is not hard to show that these are, in fact, the only elements of the n-th Spanier group. Hence

$$\ker(\pi_n(M_f, b_0) \to \check{\pi}_n(M_f, b_0)) = \{[\alpha^k * g] \mid [g] \in \pi_n(\mathbb{E}_n, b_0)\},$$

which is an uncountable subgroup.

It follows from this description that, even though M_f is not π_n-shape injective, M_f is n-homotopically Hausdorff. Indeed, it suffices to check this at the points $\alpha(t)$, $t \in I$. We give the argument for $\alpha(0) = b_0$, the other points are similar. If $0 \neq h \in \pi_n(M_f, b_0)$ has a representative in every neighborhood of b_0 in M_f, then clearly $h \in \ker(\Psi_n)$. Hence, $h = [\alpha^k * g]$ for $[g] \in \pi_n(\mathbb{E}_n, b_0)$. Since M_f retracts onto the circle parameterized by α^k, the hypothesis on h can only hold if $k = 0$. However, there is a basis of neighborhoods of b_0 in M_f that deformation retract onto an open neighborhood of b_0 in $\mathbb{E}_n$. Thus $[g]$ has a representative in every neighborhood of b_0 in $\pi_n(\mathbb{E}_n, b_0)$, giving $h = [g] \in \ker(\pi_n(\mathbb{E}_n, b_0) \to \check{\pi}_n(\mathbb{E}_n, b_0)) = 0$.

It is an important feature of Example 5.3 that M_f is not simply connected and has multiple points at which it is not semilocally π_n-trivial. This motivates the following application of Theorem 1.1, which identifies a partial converse of the implication π_n-shape injective $\Rightarrow$ n-homotopically Hausdorff.

Corollary 5.4. *Let $n \geqslant 2$ and X be a simply-connected, UV^{n-1}, compact Hausdorff space such that X fails to be semilocally n-trivial only at a single point $x \in X$. Then every element $g \in \ker(\Psi_n)$ is represented by a map with image in every neighborhood of x. In particular, if X is n-homotopically Hausdorff at x, then X is π_n-shape injective.*

Proof. According to Remark 3.5, we may take x to be the basepoint of X. Let $0 \neq g \in \ker(\Psi_n)$. By Theorem 1.1, $g \in \pi_n^{Sp}(X, x)$. Since X is compact Hausdorff, we may replace $O(X)$ by the cofinal sub-directed order $O_F(X)$ consisting of finite open covers $\mathscr{U}$ of X with the property that there is a unique $W_{\mathscr{U}} \in \mathscr{U}$ with $x \in W_{\mathscr{U}}$. For each $\mathscr{U} \in O_F(X)$, we can write $g = \prod_{i=1}^{m_{\mathscr{U}}} [\alpha_{\mathscr{U},i} * f_{\mathscr{U},i}]$ where $f_{\mathscr{U},i} : S^n \to U_{\mathscr{U},i}$ is a non-null-homotopic map for some $U_{\mathscr{U},i} \in \mathscr{U}$.

Let V be an open neighborhood of x. We check that g is represented by a map with image in V. Since X is UV^0 at x, there exists an open neighborhood V' of x such that any two points of V' may be connected by a path in V. Now, we fix $\mathscr{U}_0 \in O_F(X)$ such that $W_{\mathscr{U}_0} \subseteq V'$. Then $W_{\mathscr{V}} \subseteq V'$ whenever $\mathscr{V} \in O_F(X)$ refines $\mathscr{U}_0$.

We claim that for sufficiently refined $\mathscr{V}$, all of the maps $f_{\mathscr{V},i}$ have image in V'. Suppose, to obtain a contradiction, there is a subset $T \subseteq \{\mathscr{V} \in O_F(X) \mid \mathscr{U}_0 \preceq \mathscr{V}\}$, which is cofinal in $O_F(X)$ and such that for every $\mathscr{V} \in T$ there exists $i_{\mathscr{V}} \in \{1, 2, \ldots, m_{\mathscr{V}}\}$ such that $\mathrm{Im}(f_{\mathscr{V},i_{\mathscr{V}}}) \nsubseteq V'$. Find $y_{\mathscr{V},i_{\mathscr{V}}} \in S^n$ such that $f_{\mathscr{V},i_{\mathscr{V}}}(y_{\mathscr{V},i_{\mathscr{V}}}) \in U_{\mathscr{V},i} \backslash V' \subseteq U_{\mathscr{V},i} \backslash W_{\mathscr{U}_0}$. Since X is compact, we may replace T with a cofinal directed subset so that the net $\{f_{\mathscr{V},i_{\mathscr{V}}}(y_{\mathscr{V},i_{\mathscr{V}}})\}_{\mathscr{V} \in T}$ converges to a point $y \in X$. Let Y be an open neighborhood of y in X. Find $\mathscr{V}_0 \in O_F(X)$ such that there exists a unique neighborhood $V_0 \in \mathscr{V}_0$ with $y \in V_0$ and which also satisfies $V_0 \subseteq Y$. Then $U_{\mathscr{V}_0, i_{\mathscr{V}_0}} = V_0 \subseteq Y$. Moreover, if $\mathscr{V} \in T$ refines $\mathscr{V}_0$, then $\mathrm{Im}(f_{\mathscr{V},i_{\mathscr{V}}}) \subseteq U_{\mathscr{V},i_{\mathscr{V}}} \subseteq V_0 \subseteq Y$. However, for every $\mathscr{V}$, $f_{\mathscr{V},i_{\mathscr{V}}}$ is not null-homotopic in X. Thus, since Y represents an arbitrary neighborhood of y, X is not semilocally π_n-trivial at y. By assumption, we must have $x = y$. Since $\{f_{\mathscr{V},i_{\mathscr{V}}}(y_{\mathscr{V},i_{\mathscr{V}}})\}_{\mathscr{V} \in T}$ converges to x, the same argument where V' replaces Y shows that $\mathrm{Im}(f_{\mathscr{V},i_{\mathscr{V}}}) \subseteq V'$ for sufficiently refined $\mathscr{V} \in T$; a contradiction. Since the claim is proved, there exists $\mathscr{U}_0 \preceq \mathscr{U}_1$ in $O_F(X)$ such that whenever $\mathscr{U}_1 \preceq \mathscr{V}$, we have $\mathrm{Im}(f_{\mathscr{V},i}) \subseteq V'$ for all $i \in \{1, 2, \ldots, m_{\mathscr{V}}\}$.

Fix a refinement $\mathscr{V}$ of $\mathscr{U}_1$ in $O_F(X)$. For all $i \in \{1, 2, \ldots, m_{\mathscr{V}}\}$, we may find a path $\beta_{\mathscr{V},i} : I \to V$ from x to $f_{\mathscr{V},i}(d_0)$. Since g is simply connected, we have $[\alpha_{\mathscr{V},i} * f_{\mathscr{U},i}] = [\beta_{\mathscr{V},i} * f_{\mathscr{U},i}]$ for all i. Thus g is represented by $\prod_{i=1}^{m_{\mathscr{V}}} \beta_{\mathscr{V},i} * f_{\mathscr{V},i}$, which has image in V. ☐

Remark 5.5 (Topologies on homotopy groups)**.** Given a group G and a collection of subgroups $\{N_j \mid j \in J\}$ of G such that for all $j, j' \in J$, there exists $k \in J$ such that $N_k \subseteq N_j \cap N_{j'}$, we can generate a topology on G by taking the set

$\{gN_j \mid j \in J, g \in G\}$ of left cosets as a basis. We can apply this to both the collection of Spanier subgroups $\pi_n^{Sp}(\mathscr{U}, x_0)$ and the collection of kernels $\ker(p_{\mathscr{U}\#})$ to define two natural topologies on $\pi_n(X, x_0)$.

(1) The *Spanier topology* on $\pi_n(X, x_0)$ is generated by the left cosets of Spanier groups $\pi_n(\mathscr{U}, x_0)$ for $\mathscr{U} \in O(X)$.
(2) The *shape topology* on $\pi_n(X, x_0)$ is generated by left cosets of the kernels $\ker(p_{\mathscr{U}\#})$ where $(\mathscr{U}, U_0) \in \Lambda$. Equivalently, the shape topology is the initial topology with respect to the map Ψ_n where the groups $\pi_n(|N(\mathscr{U})|, U_0)$ are given the discrete topology and $\check{\pi}_n(X, x_0)$ is given the inverse limit topology.

Lemma 3.10 ensures the Spanier topology is always finer than the shape topology. Lemma 5.1 then implies that, whenever X is paracompact, Hausdorff, and UV^{n-1}, the two topologies agree. Moreover, $\pi_n(X, x_0)$ is Hausdorff in the shape topology if and only if X is π_n-shape injective.

6 When is Ψ_n an isomorphism?

It is a result of Kozlowski-Segal [23] that if X is paracompact Hausdorff and UV^n, then $\Psi_n : \pi_n(X, x) \to \check{\pi}_n(X, x)$ is an isomorphism. This result was first proved for compact metric spaces in [24]. The assumption that X is UV^n assumes that small maps $S^n \to X$ may be contracted by small null-homotopies. However, if $\mathbb{E}_n$ is the n-dimensional earring space, then the cone $C\mathbb{E}_n$ is UV^{n-1} but not UV^n. However, $C\mathbb{E}_n$ is contractible and so Ψ_n is clearly an isomorphism of trivial groups. Certainly, many other examples in this range exist. Our Spanier group-based approach allows us to generalize Kozlowksi-Segal's theorem in a way that includes this example by removing the need for "small" homotopies in dimension n. For simplicity, we will sometimes suppress the pointedness of open covers and simply write $\mathscr{U}$ for elements of Λ.

Lemma 6.1. *Let $n \geqslant 1$. Suppose that X is paracompact, Hausdorff, and UV^{n-1}. If $([f_{\mathscr{U}}])_{\mathscr{U} \in \Lambda} \in \check{\pi}_1(X, x_0)$, then for every $\mathscr{U} \in \Lambda$, there exists $[g] \in \pi_n(X, x)$ such that $(p_{\mathscr{U}})_{\#}([g]) = [f_{\mathscr{U}}]$.*

Proof. With $(\mathscr{U}, U_0) \in \Lambda$ and $p_{\mathscr{U}}$ fixed, consider a representing map $f_{\mathscr{U}} : (|\partial\Delta_{n+1}|, d_0) \to (|N(\mathscr{U})|, U_0)$. Let $\mathscr{U}' = \{p_{\mathscr{U}}^{-1}(\mathrm{st}(U, |N(\mathscr{U})|)) \mid U \in \mathscr{U}\}$. Since $p_{\mathscr{U}}^{-1}(\mathrm{st}(U, |N(\mathscr{U})|)) \subseteq U$ for all $U \in \mathscr{U}$, we have $\mathscr{U} \preceq \mathscr{U}'$. Applying Lemmas 4.2 and 4.6 we can choose the following sequence of refinements of $\mathscr{U}'$. First, we choose a star refinement $\mathscr{U}' \preceq_{**} \mathscr{W}$ so that for every $W \in \mathscr{W}$, there exists $U' \in \mathscr{U}'$ such that $\mathrm{St}(W, \mathscr{W}) \subseteq U'$. In this case, we can choose the projection map $p_{\mathscr{U}'\mathscr{W}} : |N(\mathscr{W})| \to |N(\mathscr{U}')|$ so that if $p_{\mathscr{U}'\mathscr{W}}(W) = U'$ on vertices, then $\mathrm{St}(W, \mathscr{W}) \subseteq U'$ in X. This choice will be important near the end of the proof.

To construct g, we must take further refinements. First, we choose a sequence of a refinements

$$\mathscr{W} = \mathscr{W}_n \preceq_* \mathscr{V}_n \preceq_*^{n-1} \mathscr{W}_{n-1} \preceq_* \cdots \preceq_*^2 \mathscr{W}_2 \preceq_* \mathscr{V}_2 \preceq_*^1 \mathscr{W}_1 \preceq_* \mathscr{V}_1 \preceq_*^0 \mathscr{W}_0$$

followed by one last refinement $\mathscr{W}_0 \preceq_* \mathscr{V}_0 = \mathscr{V}$. Let $V_0 \in \mathscr{V}$ be any set containing x_0 and recall that since X is paracompact Hausdorff $(\mathscr{V}, V_0) \in \Lambda$. For some choice of canonical map $p_{\mathscr{V}}$, we have $p_{\mathscr{V}}^{-1}(\mathrm{st}(V, N(\mathscr{V}))) \subseteq V$ for all $V \in \mathscr{V}$.

Recall that we have assumed the existence of a map $f_{\mathscr{V}} : (\partial\Delta_{n+1}, d_0) \to (|N(\mathscr{V})|, V_0)$ such that $p_{\mathscr{U}\mathscr{V}\#}([f_{\mathscr{V}}]) = [f_{\mathscr{U}}]$. Set $Y_V = f_{\mathscr{V}}^{-1}(\mathrm{st}(V, N(\mathscr{V})))$ so that $\mathscr{Y} = \{Y_V \mid V \in \mathscr{V}\}$ is an open cover of $\partial\Delta_{n+1}$. As before, we find a simplicial approximation for $f_{\mathscr{V}}$. Find $m \in \mathbb{N}$ such that the star $\mathrm{st}(a, \mathrm{sd}^m \partial\Delta_{n+1})$ of each vertex a in $\mathrm{sd}^m\partial\Delta_{n+1}$ lies in a set $Y_{V_a} \in \mathscr{Y}$ for some $V_a \in \mathscr{V}$. Since $f_{\mathscr{V}}(d_0) = V_0$, we may take $V_{d_0} = V_0$. The assignment $a \mapsto V_a$ on vertices extends to a simplicial approximation $f' : \mathrm{sd}^m\partial\Delta_{n+1} \to |N(\mathscr{V})|$ of $f_{\mathscr{V}}$, i.e. a simplicial map f' such that

$$f_{\mathscr{V}}(\mathrm{st}(a, \mathrm{sd}^m\partial\Delta_{n+1})) \subseteq \mathrm{st}(f'(a), |N(\mathscr{V})|) = \mathrm{st}(V_a, |N(\mathscr{V})|)$$

for each vertex a.

We begin to define g with the constant map $\{d_0\} \to X$ sending d_0 to x_0. In preparation for applications of Lemma 4.7, set $K = \mathrm{sd}^m\partial\Delta_{n+1}$ and $L = \{d_0\}$ so that $K[k] = K_k$. First, we define a map $g_0 : M[0] \to X$ by picking, for each vertex $a \in K_0$, a point $g_0(a) \in V_a$. In particular, set $g_0(d_0) = x_0$. This choice is well defined since we have $p_{\mathscr{V}}(x_0) = V_0 \in \mathrm{st}(V_{d_0}, N(\mathscr{V}))$ and thus $g_0(d_0) = x_0 \in p_{\mathscr{V}}^{-1}(\mathrm{st}(V_{d_0}, N(\mathscr{V}))) \subseteq V_{d_0}$. Note that f' maps every simplex $\sigma = [a_0, a_1, \ldots, a_k]$ of K to the simplex of $|N(\mathscr{V})|$ spanned by $\{V_{a_i} \mid 0 \leqslant i \leqslant k\}$. By definition of the nerve, we have $\bigcap\{V_{a_i} \mid 0 \leqslant i \leqslant k\} \neq \varnothing$. Pick a point $x_\sigma \in \bigcap\{V_{a_i} \mid 0 \leqslant i \leqslant k\}$. By our initial choice of refinements, we have $\mathscr{U}_0 \preceq_* \mathscr{V}$. If $\sigma = [a_0, a_1, \ldots, a_n]$ is a n-simplex of K, then $\mathrm{St}(x_\sigma, \mathscr{V}) \subseteq U_{0,\sigma}$ for some $U_{0,\sigma} \in \mathscr{U}_0$. In particular $\{g_0(a_i) \mid 0 \leqslant i \leqslant n+1\} \subseteq \bigcup\{V_{a_i} \mid 0 \leqslant i \leqslant n\} \subseteq U_{0,\sigma}$. Thus g_0 maps the 0-skeleton of σ into $U_{0,\sigma}$. If $d_0 \in \sigma$, then $g_0(d_0) \in p_{\mathscr{V}}^{-1}(\mathrm{st}(V_{d_0}, N(\mathscr{V}))) \subseteq V_{d_0} \subseteq U_{0,\sigma}$. Hence, for every n-simplex σ of K, we have $g_0(\sigma \cap M[0]) \subseteq U_{0,\sigma}$.

We are now in the situation to recursively apply Lemma 4.7. This is similar to the application in the proof of Lemma 5.1 with the dimension $n+1$ shifted down by one so we omit the details. We obtain an extension $g : K = M[n] \to X$ of g_0 such that for every n-simplex σ of K, we have $g(\sigma) \subseteq W_\sigma$ for some $W_\sigma \in \mathscr{W} = \mathscr{U}_n$.

With g defined, we seek show that $f_{\mathscr{U}} \simeq p_{\mathscr{U}} \circ g$. Since $f' \simeq f_{\mathscr{V}}$ (by simplicial approximation), $p_{\mathscr{U}\mathscr{V}} \simeq p_{\mathscr{U}\mathscr{U}'} \circ p_{\mathscr{U}'\mathscr{W}} \circ p_{\mathscr{W}\mathscr{V}}$ (for any choice of projection maps), and $p_{\mathscr{U}\mathscr{V}} \circ f_{\mathscr{V}} \simeq f_{\mathscr{U}}$ (for any choice of projection $p_{\mathscr{U}\mathscr{V}}$), it suffices to show that $p_{\mathscr{U}\mathscr{U}'} \circ p_{\mathscr{U}'\mathscr{W}} \circ p_{\mathscr{W}\mathscr{V}} \circ f' \simeq p_{\mathscr{U}} \circ g$. We do this by proving that the simplicial map $F = p_{\mathscr{U}\mathscr{U}'} \circ p_{\mathscr{U}'\mathscr{W}} \circ p_{\mathscr{W}\mathscr{V}} \circ f' : K \to |N(\mathscr{U})|$ is a simplicial approximation for $p_{\mathscr{U}} \circ g$. Recall that this can be done by verifying the "star-condition" $p_{\mathscr{U}} \circ g(\mathrm{st}(a, K)) \subseteq \mathrm{st}(F(a), |N(\mathscr{U})|)$ for any vertex $a \in K$ [28, Ch.2 §14]. Since $n \geqslant 1$, we have $\mathscr{W} \preceq_{**} \mathscr{V}$. Hence, just like our choice of $p_{\mathscr{U}'\mathscr{W}}$, we may choose $p_{\mathscr{W}\mathscr{V}}$ so that whenever $p_{\mathscr{W}\mathscr{V}}(V) = W$, then $\mathrm{St}(V, \mathscr{V}) \subseteq W$. Also, we choose $p_{\mathscr{U}\mathscr{U}'}$ to map $p_{\mathscr{U}}^{-1}(\mathrm{st}(U, |N(\mathscr{U})|)) \mapsto U$ on vertices.

Fix a vertex $a_0 \in K$. To check the star-condition, we'll check that $p_{\mathscr{U}} \circ g(\sigma) \subseteq \mathrm{st}(F(a_0), |N(\mathscr{U})|)$ for each n-simplex σ having a_0 as a vertex. Pick an n-simplex $\sigma = [a_0, a_1, \ldots, a_n] \subseteq K$ having a_0 as a vertex. Recall that $f'(a_i) = V_{a_i}$ for each i. Set $p_{\mathscr{W}\mathscr{V}}(V_{a_i}) = W_i$ and $p_{\mathscr{U}'\mathscr{W}}(W_i) = p_{\mathscr{U}}^{-1}(\mathrm{st}(U_i, |N(\mathscr{U})|)) \in \mathscr{U}'$ for some $U_i \in \mathscr{U}$. Then $F(a_i) = U_i$ for all i. It now suffices to check that $p_{\mathscr{U}} \circ g(\sigma) \subseteq \mathrm{st}(U_0, |N(\mathscr{U})|)$. Recall that by our choice of $p_{\mathscr{U}'\mathscr{W}}$, we have $\mathrm{St}(W_0, \mathscr{W}) \subseteq p_{\mathscr{U}}^{-1}(\mathrm{st}(U_0, |N(\mathscr{U})|))$. Thus it is enough to check that $g(\sigma) \subseteq \mathrm{St}(W_0, \mathscr{W})$. By construction of g, we have $g(\sigma) \subseteq W_\sigma$ for some $W_\sigma \in \mathscr{W}$. Since $g(a_0) \in W_0 \cap W_\sigma$, we have $g(\sigma) \subseteq W_\sigma \subseteq \mathrm{St}(W_0, \mathscr{W})$, completing the proof. □

Finally, we prove our secondary result, Theorem 1.2.

Proof of Theorem 1.2. Since X is paracompact, Hausdorff, UV^{n-1}, we have $\pi_n^{Sp}(X, x_0) = \ker(\Psi_n)$ by Theorem 1.1. Since X is semilocally π_n-trivial, we have $\pi_n^{Sp}(\mathscr{U}, x_0) = 1$ for some $\mathscr{U} \in \Lambda$. It follows that Ψ_n is injective. Moreover, by Lemma 5.1, we may find $\mathscr{V} \in \Lambda$ with $\ker(p_{\mathscr{V}\#}) \subseteq \pi_n^{Sp}(\mathscr{U}, x_0)$. Thus $p_{\mathscr{V}\#} : \pi_n(X, x_0) \to \pi_n(|N(\mathscr{V})|, V_0)$ is injective. Let $([f_{\mathscr{U}}])_{\mathscr{U}\in\Lambda} \in \check{\pi}_n(X, x_0)$. By Lemma 6.1, for each $\mathscr{U} \in \Lambda$, there exists $[g_{\mathscr{U}}] \in \pi_n(X, x_0)$ such that $p_{\mathscr{U}}([g_{\mathscr{U}}]) = [f_{\mathscr{U}}]$. If $\mathscr{V} \preceq \mathscr{W}$, then we have

$$p_{\mathscr{V}\#}([g_{\mathscr{V}}]) = [f_{\mathscr{V}}] = p_{\mathscr{V}\mathscr{W}\#}([f_{\mathscr{W}}]) = p_{\mathscr{V}\mathscr{W}\#} \circ p_{\mathscr{W}\#}([g_{\mathscr{W}}]) = p_{\mathscr{V}\#}([g_{\mathscr{W}}])$$

Since $p_{\mathscr{V}\#}$ is injective, it follows that $[g_{\mathscr{W}}] = [g_{\mathscr{V}}]$ whenever $\mathscr{V} \preceq \mathscr{W}$. Setting $[g] = [g_{\mathscr{V}}]$ gives $\Psi_n([g]) = ([f_{\mathscr{U}}])_{\mathscr{U}\in\Lambda}$. Hence, Ψ_n is surjective. □

7 Examples

Example 7.1. Fix $n \geqslant 2$. When X is a metrizable UV^{n-1} space, the cone CX and unreduced suspension SX are UV^{n-1} and semilocally π_n-trivial but need not be UV^n. This occurs in the case $X = \mathbb{E}_n$ or if $X = Y \vee \mathbb{E}_n$ where Y is a CW-complex. In such cases, $\Psi_n : \pi_n(SX) \to \check{\pi}_n(SX)$ is an isomorphism. One point unions of such cones and suspensions, e.g. $CX \vee CY$ or $CX \vee SY$ also meet the hypotheses of Theorem 1.2 (checking this is fairly technical [3]) but need not be UV^n.

Example 7.2. The converse of Theorem 1.2 does not hold. For $n \geqslant 2$, $\mathbb{E}_n$ is UV^{n-1} but is not semilocally π_n-trivial at the wedgepoint x_0. However, $\Psi_n : \pi_n(\mathbb{E}_n, x_0) \to \check{\pi}_n(\mathbb{E}_n, x_0)$ is an isomorphism where both groups are canonically isomorphic to $\mathbb{Z}^{\mathbb{N}}$ [12]. Additionally, for the infinite direct product $\prod_{\mathbb{N}} S^n$, $\Psi_k : \pi_k(\prod_{\mathbb{N}} S^n, x_0) \to \check{\pi}_k(\prod_{\mathbb{N}} S^n, x_0)$ is an isomorphism for all $k \geqslant 1$ even though $\prod_{\mathbb{N}} S^n$ is not UV^{k-1} when $k - 1 \geqslant n$.

Example 7.3. We can also modify the mapping torus M_f from Example 5.3 so that Ψ_n becomes an isomorphism (recall that $n \geqslant 2$ is fixed). Let $X = M_f \cup C\mathbb{E}_n$ be the mapping cone of the inclusion $\mathbb{E}_n \to M_f$. For the same reason M_f is UV^{n-1}, the space X is UV^{n-1}. Moreover, if U is a neighborhood of $\alpha(t)$ that deformation retracts onto a homeomorphic copy of $\mathbb{E}_n$, then any map $S^n \to U$ may be freely homotoped "around" the torus and into the cone. It follows that X is semilocally π_n-trivial. We conclude from Theorem 1.2 that $\Psi_n : \pi_n(X) \to \check{\pi}_n(X)$ is an isomorphism. Since sufficiently fine covers of X

always give nerves homotopy equivalent to $S^1 \vee S^{n+1}$, we have $\check{\pi}_n(X, b_0) = 0$. Thus $\pi_n(X) = 0$.

Example 7.4. Let $n \geqslant 2$ and $X = \mathbb{E}_1 \vee S^n$ (see Figure 2). Note that because $\mathbb{E}_1$ is aspherical [6, 8], X is semilocally π_n-trivial. However, X is not UV^1 because it has $\mathbb{E}_1$ as a retract. It is shown in [3] that $\pi_n(X) \cong \bigoplus_{\pi_1(\mathbb{E}_1)} \pi_n(S^n) \cong \bigoplus_{\pi_1(\mathbb{E}_1)} \mathbb{Z}$ and that $\Psi_n : \pi_n(X) \to \check{\pi}_n(X)$ is injective. In particular, we may represent elements of $\pi_n(X)$ as finite-support sums $\sum_{\beta \in \pi_1(\mathbb{E}_1)} m_\beta$ where $m_\beta \in \mathbb{Z}$. We show that Ψ_n is not surjective.

Identify $\pi_1(X)$ with $\pi_1(\mathbb{E}_1)$ and recall from [9] that we can represent the elements of $\pi_1(\mathbb{E}_1)$ as countably infinite reduced words indexed by a countable linear order (with a countable alphabet $\beta_1, \beta_2, \beta_3, \dots$). Here β_j is represented by a loop $S^1 \to \mathbb{E}_1$ going once around the j-th circle. Let X_j be the union of S^n and the largest j circles of $\mathbb{E}_1$ so that $X = \varprojlim_j X_j$. Identify $\pi_1(X_j)$ with the free group F_j on generators $\beta_1, \beta_2, \dots \beta_j$ and note that $\pi_n(X_j) \cong \bigoplus_{F_j} \mathbb{Z}$. Thus we may view an element of $\pi_n(X_j)$ as a finite-support sums $\sum_{w \in F_j} m_w$ of integers indexed over reduced words in F_j. Let $d_{j+1,j} : F_{j+1} \to F_j$ be the homomorphism that deletes the letter β_{j+1}. Consider the inverse limit $\check{\pi}_1(X) = \varprojlim_j (F_j, d_{j+1,j})$. The map $X \to X_j$ that collapses all but the first j-circles of $\mathbb{E}_1$ induces a homomorphism $d_j : \pi_1(X) \to F_j$. There is a canonical homomorphism $\phi : \pi_1(X) \to \check{\pi}_1(X) = \varprojlim_j (F_j, d_{j+1,j})$ given by $\phi(\beta) = (d_1(\beta), d_2(\beta), \dots)$, which is known to be injective [27] but not surjective. For example, if $x_k = \prod_{j=1}^{k} [\beta_1, \beta_j]$, then $(x_1, x_2, x_3, x_4, \dots)$ is an element of $\check{\pi}_1(X)$ not in the image of ϕ.

The bonding map $b_{j+1,j} : \pi_n(X_{j+1}) \to \pi_n(X_j)$ sends a sum $\sum_{w \in F_{j+1}} m_w$ to $\sum_{v \in F_j} p_v$ where $p_v = \sum_{d_{j+1,j}(w) = v} m_w$. Similarly, projection map $b_j : \pi_n(X) \to \pi_n(X_j)$ sends the sum $\sum_{\beta \in \pi_1(X)} n_\beta$ to $\sum_{v \in F_j} m_v$ where $m_v = \sum_{d_j(\beta) = v} m_\beta$. Let $y_j \in \pi_n(X)$ be the sum whose only non-zero coefficient is the x_j-coefficient, which is 1. Since $d_{j+1,j}(x_{j+1}) = x_j$, it's clear that $(y_1, y_2, y_3, \dots) \in \check{\pi}_n(X)$. Suppose $\Psi_n(\sum_\beta m_\beta) = (y_1, y_2, y_3, \dots)$. Writing $\sum_\beta m_\beta$ as a finite sum $\sum_{i=1}^{r} m_{\beta_i}$ for non-zero m_{β_i}, we must have $\sum_{d_j(\beta_i) = x_j} m_{\beta_i} = 1$ for all $j \in \mathbb{N}$. Since there are only finitely many β_i involved, there must exist at least one i for which $d_j(\beta_i) = x_j$ for infinitely many j. For such i, we have $\phi(\beta_i) = (x_1, x_2, x_3, \dots)$, which, as mentioned above, is impossible. Hence Ψ_n is not surjective.

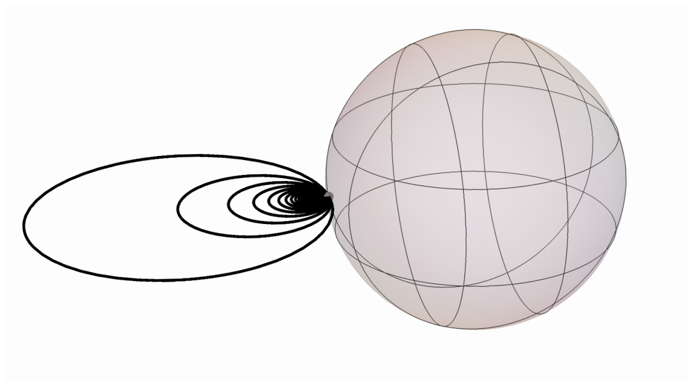

Figure 2: The one point union $\mathbb{E}_1 \vee S^2$.

The previous example shows why we cannot do away with the UV^{n-1} hypothesis in Theorem 1.2. Since we weakened the hypothesis from [23] in dimension n and no hypothesis in dimension n is required for Theorem 1.1, one might suspect that we might be able to do away with the dimension n hypothesis completely. The next example, which is a higher analogue of the harmonic archipelago [2, 7, 20] shows why this is not possible.

Example 7.5. Let $n \geqslant 2$ and $\ell_j : S^n \to \mathbb{E}_n$ be the inclusion of the j-th n-sphere in $\mathbb{E}_n$. Let X be the space obtained by attaching $(n+1)$-cells to $\mathbb{E}_n$ using the attaching maps ℓ_j. Since $\mathbb{E}^n$ is UV^{n-1} it follows easily that X is UV^{n-1}. However, X is not semilocally π_n-trivial at the wedgepoint x_0 of $\mathbb{E}^n$. Indeed, the infinite concatenation maps $\prod_{j \geqslant k} \ell_j = \ell_k \cdot \ell_{k+1} \cdots$ are not null-homotopic (using a standard argument that works for the harmonic archipelago) but are all homotopic to each other. Thus $\pi_n(X, x_0) \neq 0$. However for sufficiently fine open covers $\mathscr{U} \in O(X)$, $|N(\mathscr{U})|$ is homotopy equivalent to a wedge of $(n+1)$-spheres and is therefore n-connected. Thus $\check{\pi}_n(X, b_0) = 0$. Thus, despite X being UV^{n-1}, Ψ_n is not an isomorphism. In fact, $\pi_n(X, x_0) = \pi_n^{Sp}(X, x_0) = \ker(\Psi_n)$. The reader might also note that since $\mathbb{E}^{n-1}$ is $(n-1)$-connected and $\pi_n(\mathbb{E}_n) \cong H_n(\mathbb{E}_n) \cong \mathbb{Z}^{\mathbb{N}}$, X will also be $(n-1)$-connected. A Meyer-Vietoris Sequence argument similar to that in [20] can then be used to show $\pi_n(X, x_0) \cong H_n(X) \cong \mathbb{Z}^{\mathbb{N}} / \bigoplus_{\mathbb{N}} \mathbb{Z}$.

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