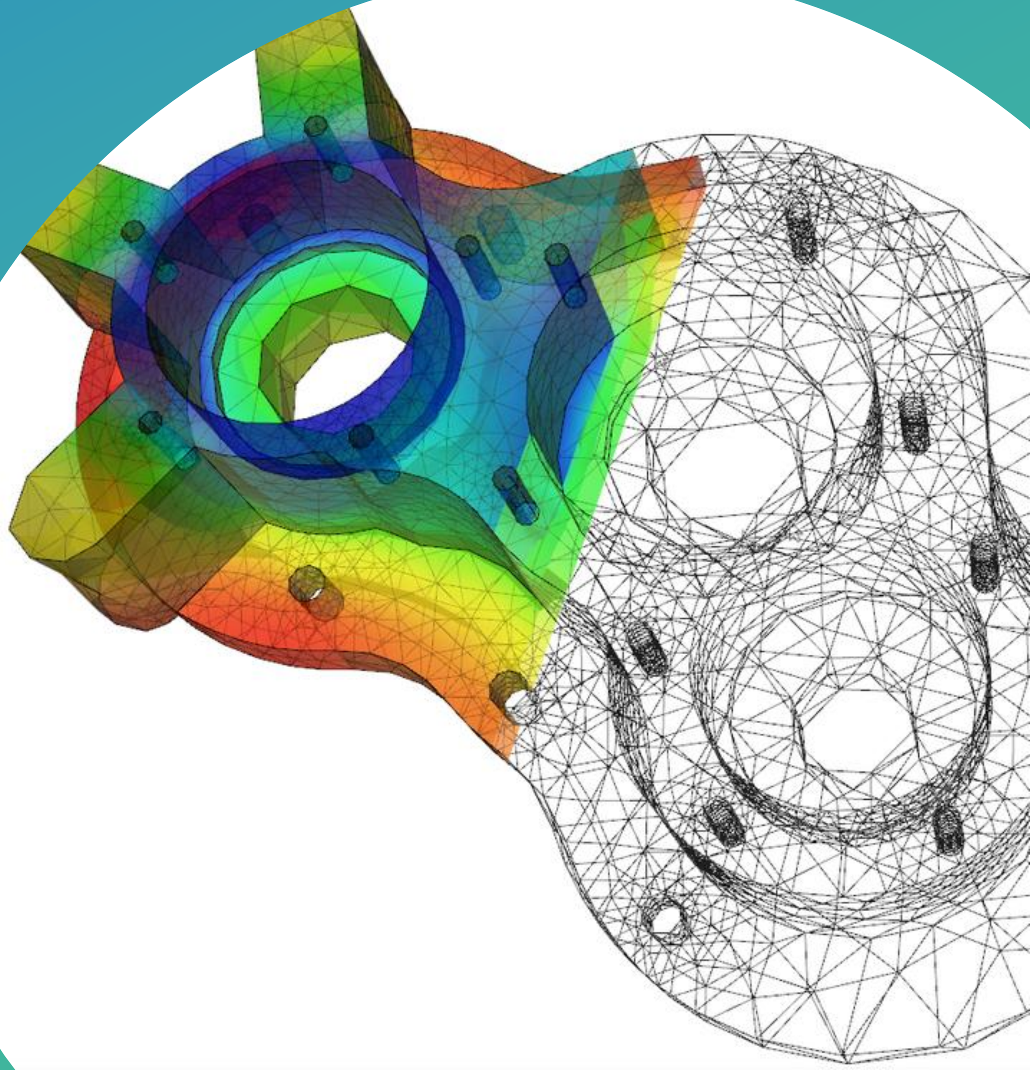
SOLVING THE HEAT EQUATION WITH INTERFACES

WCUPA
RESEARCH AND CREATIVE ACTIVITY DAY
APRIL 29, 2021



PRESENTER: MICHAEL BAUER
WITH: REX LLEWELLYN, SHAUNA FRANK
MENTORED BY: DR. CHUAN LI

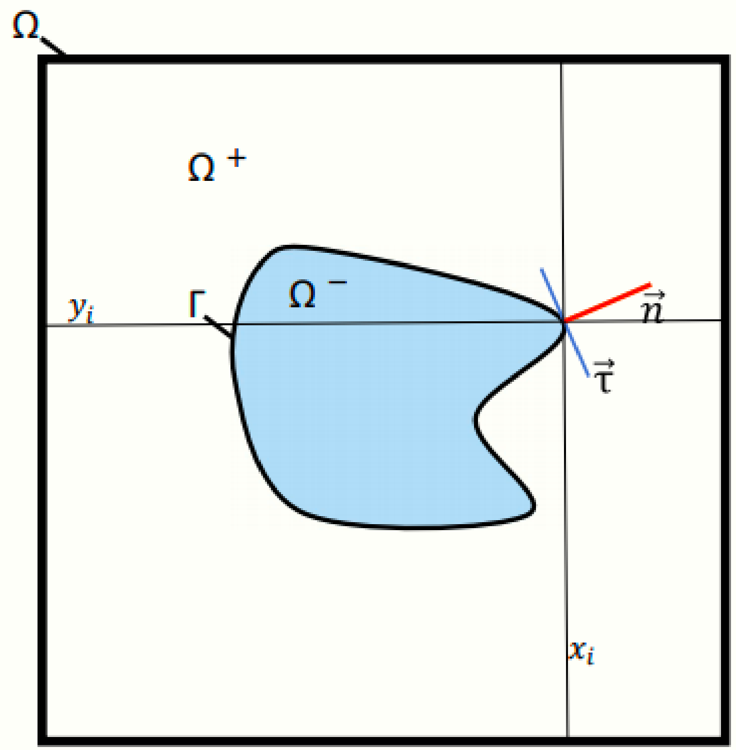
Solving the Heat Equation with Interfaces

$$u(x,y,t) = \begin{cases} u^+(x,y,t) & \text{in } \Omega^+ \\ u^-(x,y,t) & \text{in } \Omega^- \end{cases}$$

$$\frac{\partial u}{\partial t} = \nabla(\alpha \nabla u) + f$$

$$[u] = u^+ - u^- = \phi$$

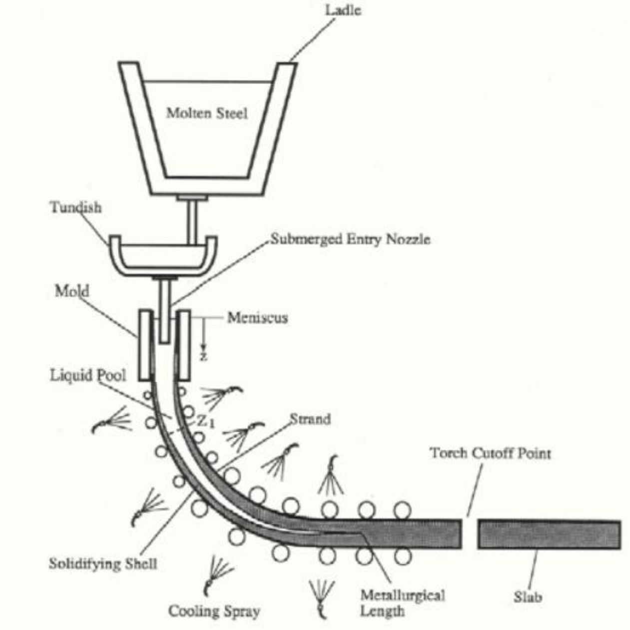
$$[\alpha u_n] = \alpha^+ \frac{\partial u^+}{\partial n} - \alpha^- \frac{u^-}{\partial n} = \psi$$



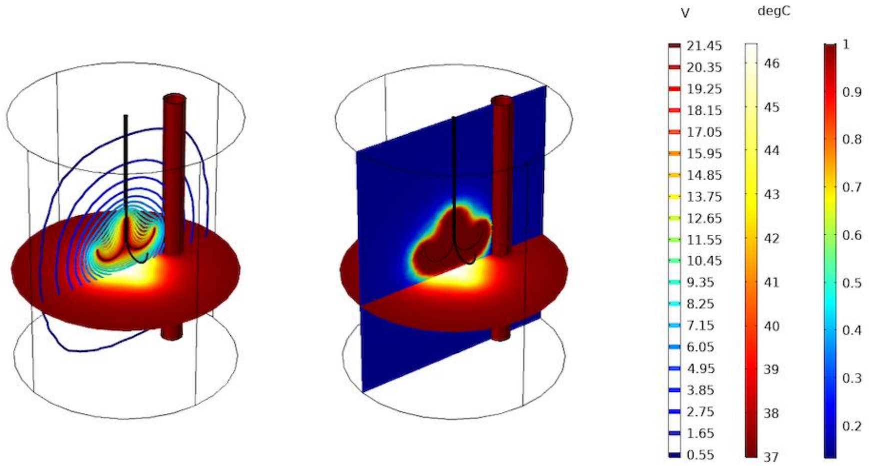
- α – diffusion coefficient
- u – function of interest
- Ω – domain of interest
- Γ – interface
- Φ – zeroth jump condition
- ψ – first jump condition

Applications

- Metallurgy
  - Steel Continuous Casting

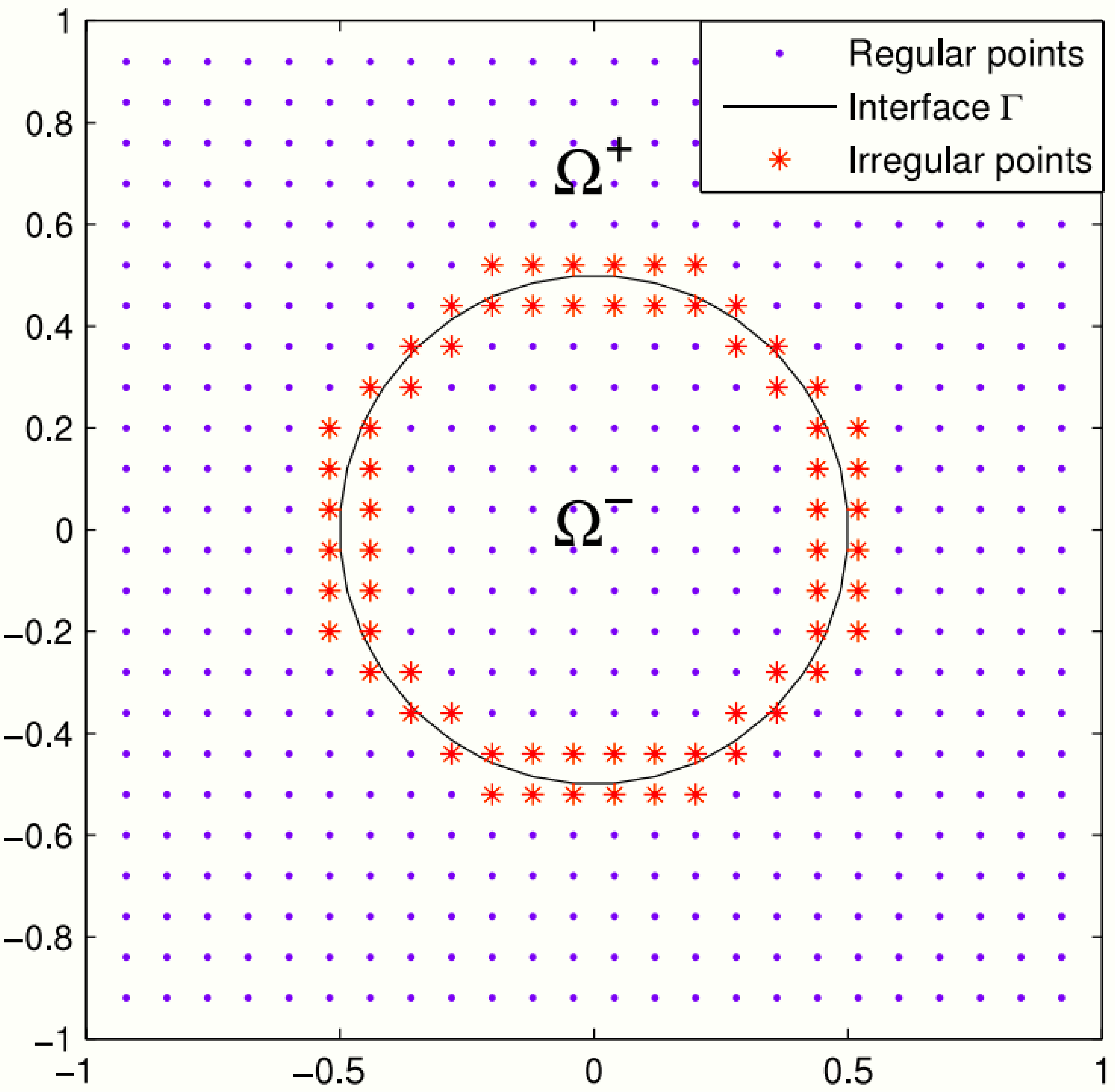


- Mathematical Biology
  - Cancer Treatment
  - Ecological Modeling



Numerical Treatment

$$u_{i,j}^k = u(x_i, y_j, t_k)$$



Temporal Discretization

- Euler Method
  - 1st Order Accuracy

$$\frac{u_{i,j}^{k+1} - u_{i,j}^{k}}{\alpha \Delta t} = \delta_{xx} u_{i,j}^{k+1} + \frac{f_{i,j}^{k+1}}{\alpha}$$

Spatial Discretization

At regular nodes

$$\delta_{xx} u_{i,j}^{k+1} = \frac{1}{\Delta x^2}(u_{i-1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i+1,j}^{k+1})$$

At nodes adjacent to interface

$$\delta_{xx} u_{i,j}^{k+1} = \frac{1}{\Delta x^2}(\tilde{u}_{i-1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i+1,j}^{k+1})$$

$$\delta_{xx} u_{i,j}^{k+1} = \frac{1}{\Delta x^2}(u_{i-1,j}^{k+1} - 2u_{i,j}^{k+1} + \tilde{u}_{i+1,j}^{k+1})$$

Spatial Discretization

Transformations

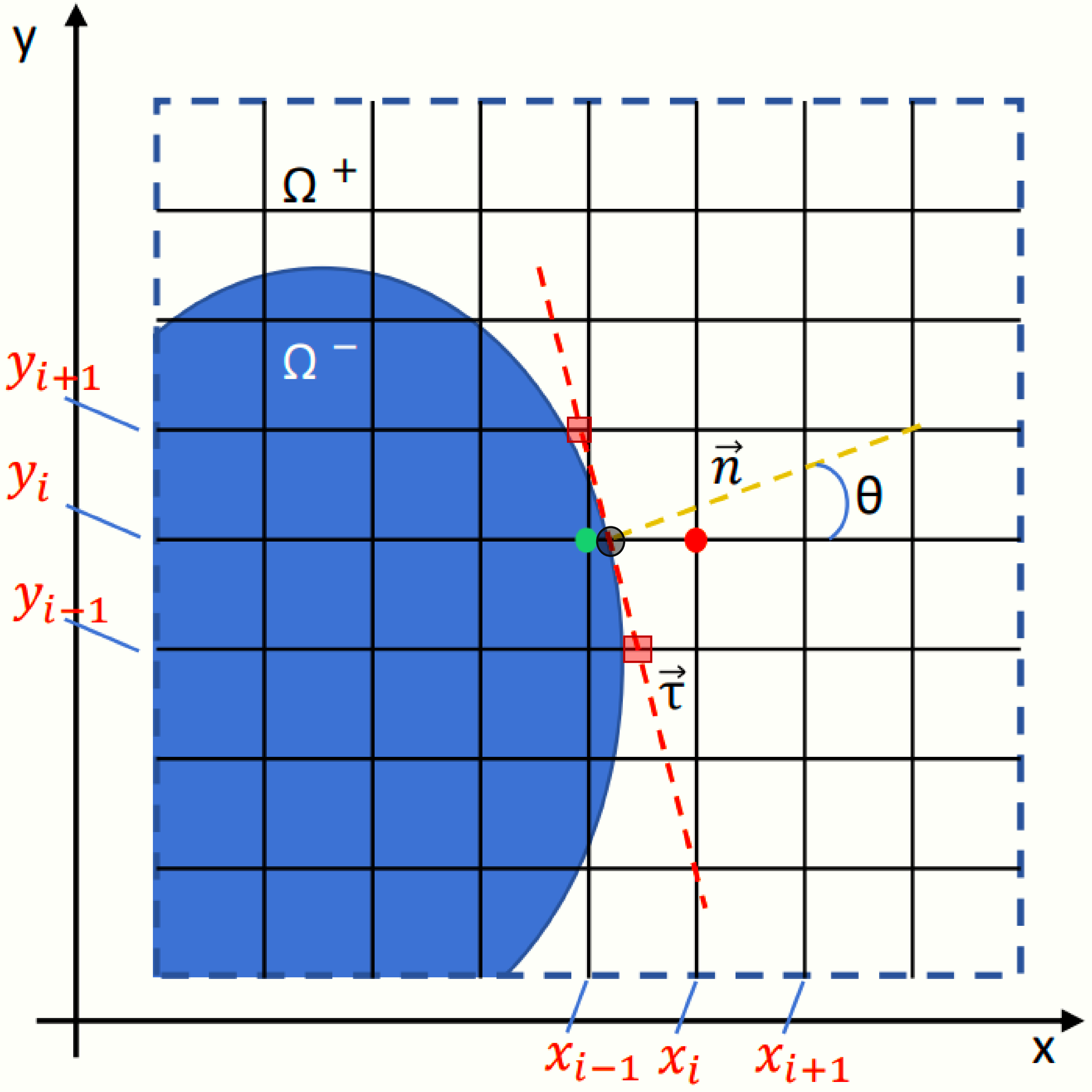
$$\frac{\partial}{\partial n} = \cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \tau} = -\sin(\theta)\frac{\partial}{\partial x} + \cos(\theta)\frac{\partial}{\partial y}$$

Jump Conditions

$$[u] = u^{+} - u^{-} = \phi \qquad [u_\tau] = \frac{\partial \phi}{\partial \tau} = \phi_\tau$$

$$[\alpha u_x] = \psi\cos(\theta) - \sin(\theta)(\alpha^{+} - \alpha^{-})u_\tau^{+} - \sin(\theta)\alpha^{-}\phi_\tau = \bar{\psi}$$



y
Ω^+
Ω^-
y_{i+1}
y_i
y_{i-1}
$\vec{n}$
θ
$\vec{\tau}$
x_{i-1}
x_i
x_{i+1}
x

Spatial Discretization

From ϕ we have

$$w^{+}_{0,1}\tilde{u}^{k+1}_{i,j} + w^{+}_{0,2}u^{k+1}_{i+1,j} + w^{+}_{0,2}u^{k+1}_{i+2,j} =$$

$$w^{-}_{0,1}u^{k+1}_{i-1,j} + w^{-}_{0,2}u^{k+1}_{i,j} + w^{-}_{0,3}\tilde{u}^{k+1}_{i+1,j} + \phi$$

From $\bar{\psi}$ we have

$$\alpha^{+}(w^{+}_{0,1}\tilde{u}^{k+1}_{i,j} + w^{+}_{0,2}u^{k+1}_{i+1,j} + w^{+}_{0,2}u^{k+1}_{i+2,j}) =$$

$$\alpha^{-}(w^{-}_{0,1}u^{k+1}_{i-1,j} + w^{-}_{0,2}u^{k+1}_{i,j} + w^{-}_{0,3}\tilde{u}^{k+1}_{i+1,j}) + \bar{\psi}$$

Numerical Experiments

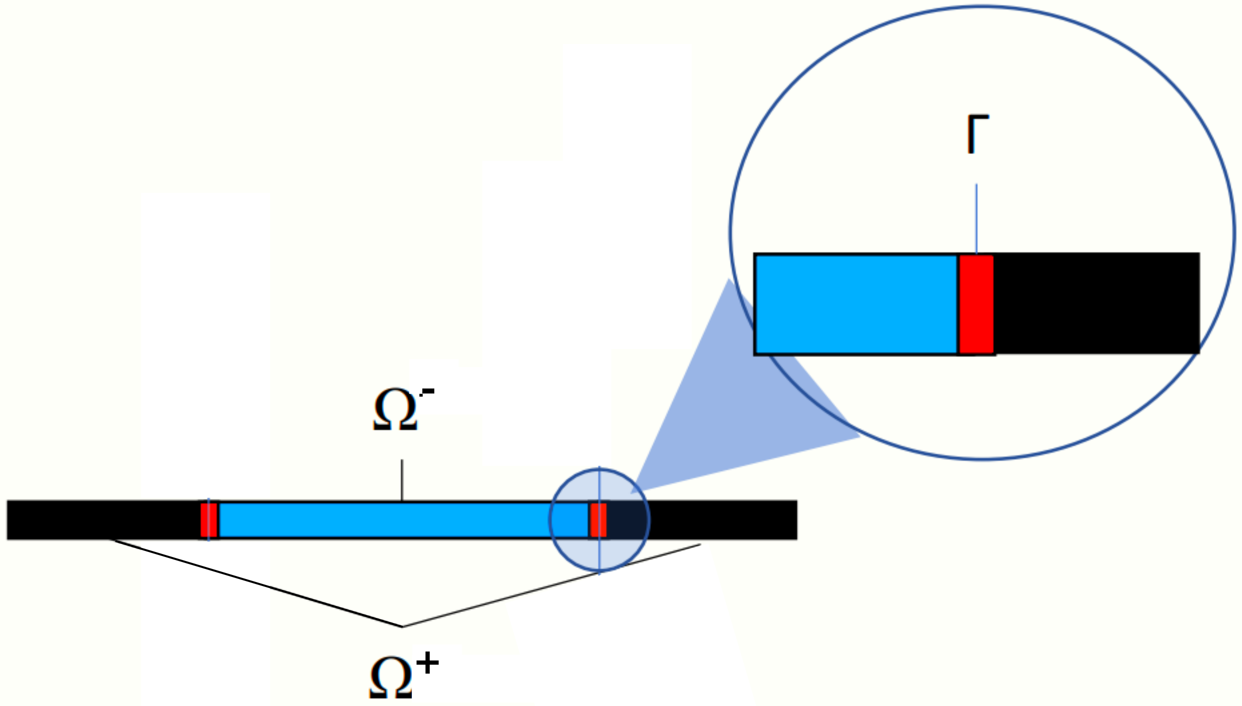
$$\frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial t^2} + f(x,t)$$

in $\Omega = \Omega^+ \cup \Omega^-$

Jump Conditions

$$[u] = u^+ - u^- = \Phi$$

$$[\alpha u_x] = \alpha^+ \frac{\partial u^+}{\partial x} - \alpha^- \frac{\partial u^-}{\partial x} = \psi$$



- α – diffusion coefficient
- u – function of interest
- Ω – domain of interest
- Γ – interface
- Φ – function jump condition
- ψ – flux jump condition

Numerical Experiments

We will look at the example where the analytical solution is given as

$$u^+(x,t) = \cos(x)\sin(t)$$

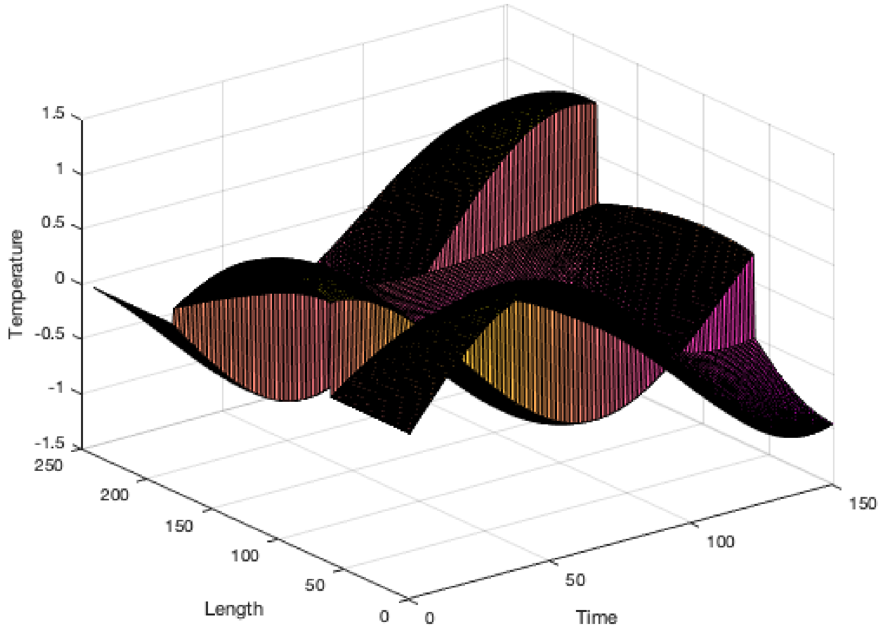
$$u^-(x,t) = \sin(x)\cos(t)$$

With the following source terms

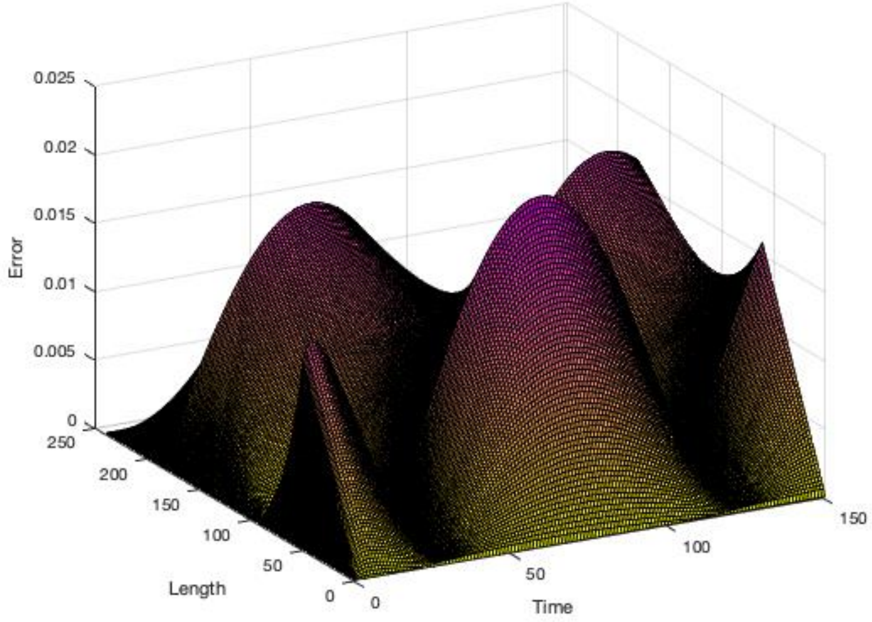
$$f^+(x,t) = \alpha^+ \cos(x)\sin(t) + \cos(x)\cos(t)$$

$$f^-(x,t) = \alpha^- \sin(x)\cos(t) - \sin(x)\sin(t)$$

Numerical Experiment



Result



Error

Numerical Experiments

We will look at the example where the analytical solution is given as

$$u^+(x,t) = e^{-t}\sin(x)$$

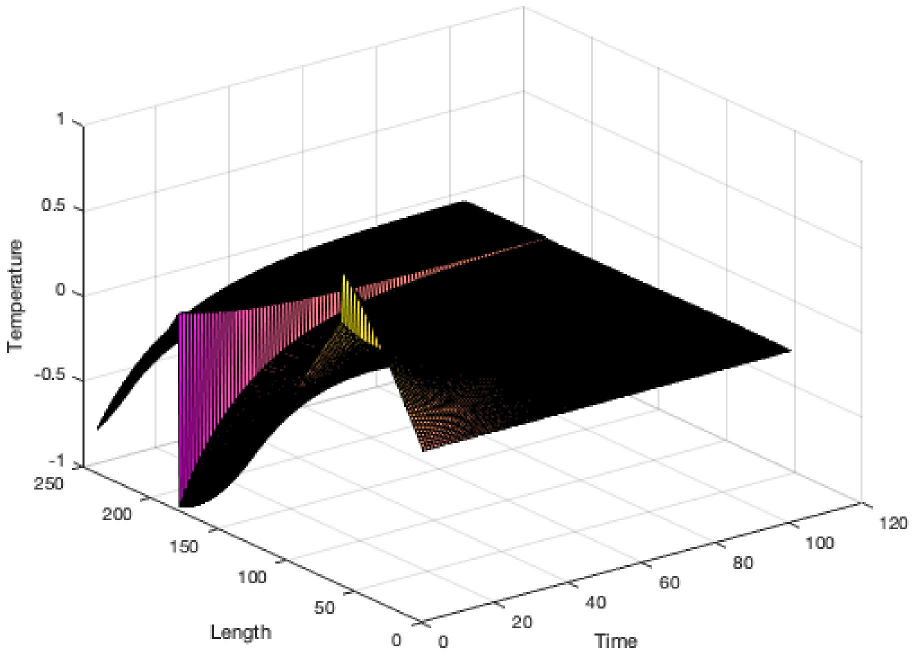
$$u^-(x,t) = e^{-t}\cos(x)$$

With the following jump conditions

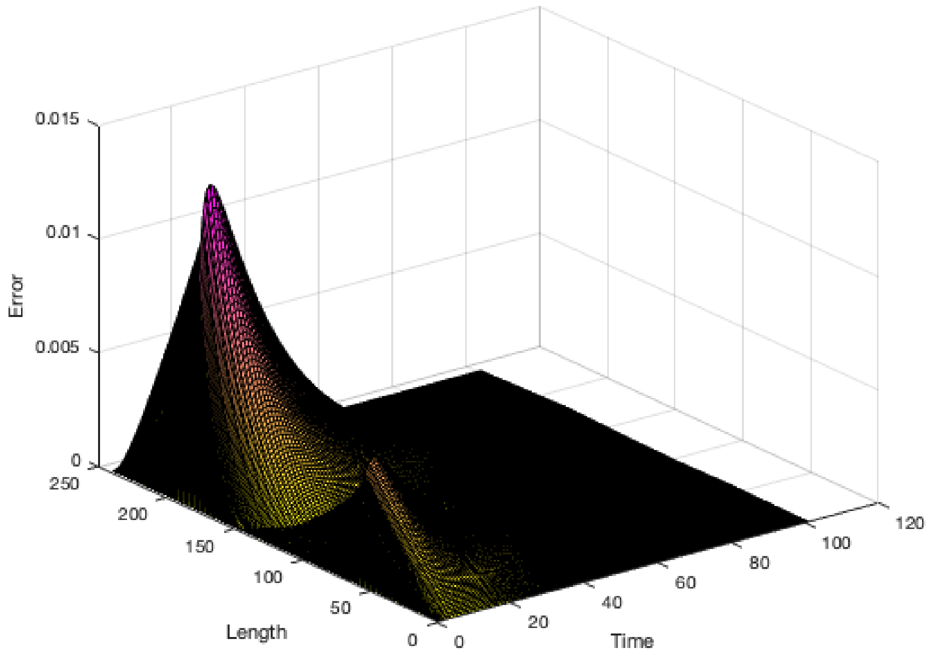
$$\phi = e^{-t}\sin(x) - e^{-t}\cos(x)$$

$$\psi = 10e^{-t}\cos(x) + e^{-t}\sin(x)$$

Numerical Experiment



Result



Error

Future Improvements

- Increase to 2D
  - This will introduce additional complications at nodes near interfaces
- Increase complexity of Interfaces
- Utilize Peaceman-Rachford method
  - Higher accuracy in temporal discretization
- Address corner cases with irregular interface geometries

Conclusion

- This exercise demonstrates the effectiveness of MIB in solving the Heat Equation with Interfaces
- Further improvements are required for applications to real-world tasks